Klassieke Mechanica b

Fall 2013

Helmut Schiessel

1 Lagrangian mechanics

1.1 Introduction

In this chapter we study dynamics in an altogether different manner. Instead of a local equation (mass $\times$ acceleration = force) we will formulate a general principle for the whole motion between two points in space. This formulation shares similarities with a law in optics, Fermat's principle (Pierre de Fermat, 1662; early version by Hero of Alexandria, c. 60): "The path of a ray of light between two points is the one that is crossed in the shortest time span". Also in optics we can formulate a local law, the Snell's law (Willebrord Snellius, 1621), in order to calculate the path of the light that it takes to minimize its optical length (another example: combination of running and swimming to reach a target inside a lake).

We give now a specific example. Minimize the time to get from A to B, with velocity c in area I and c/n in area II (Fig. 1). The total time is given by:

$$T = \frac{\sqrt{x^2 + a^2}}{c} + \frac{\sqrt{(L-x)^2 + a^2}}{c/n}.$$

The derivative of T with respect x needs to vanish:

$$\frac{x}{\sqrt{x^2 + a^2}} = \frac{n\,(L-x)}{\sqrt{(L-x)^2 + a^2}}.$$

This is nothing but Snell's law for refraction, $\sin i = n \sin r$ (first formulated by Ibn Sahl at Baghdad court in 984); zone I is here vacuum with a refractive index 1, zone II has a refractive index $n > 1$.

Fermat's principle does not mean a "deterministic" view. The light ray that starts at A does not yet "know" that it will pass through B. A better picture is given by Huygens principle (Christiaan Huygens, 1678) where a light wave is determined at any subsequent time by the sum of the secondary waves. The interference of the secondary light waves is constructive only if the phases do not vary for small deviations from the the path. Fermat's principle is then a direct consequence of that. A similar principle is at work in quantum mechanics where one can work locally (Schrödinger equation) or globally (Feynman path integrals).

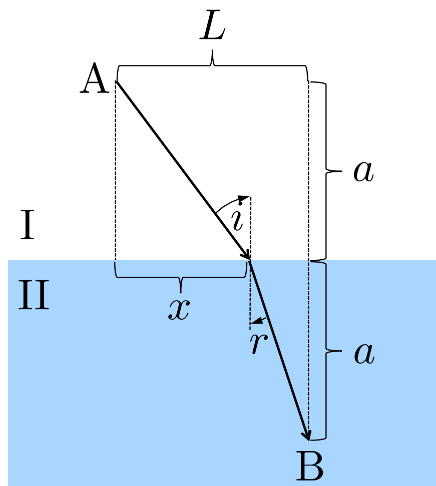


Figure 1: Snell's law for refraction.

1.2 Hamilton's variational principle

In this section we will introduce a general principle that governs the dynamics of mechanical systems. Let us first, however, recapitulate what we have learned in KMa for the case of a particle of mass m in one dimension. Its position at time t is given by $x(t)$. Assume that the particle feels a time-dependent force $f(t)$. *Newton's second law* states that the particle's mass m times its acceleration, $\ddot{x}(t) = d^2x(t)/dt^2$, equals that force:

$$m\ddot{x}(t) = f(t). \tag{1}$$

This is its *equation of motion.* As a special case of Eq. 1 consider a particle in an external potential $V(x)$. In that case $f(t) = -dV(x(t))/dx$ and hence

$$m\ddot{x}(t) = -\frac{dV(x(t))}{dx}. \tag{2}$$

We introduce now Hamilton's principle (Sir William Rowan Hamilton, 1834) which states that the dynamics of such a physical system is determined by a variational principle. As the first step we write down the *Lagrangian* [Lagraniaan] L of the system that is given by the kinetic minus the potential energy. For the particle in the potential this leads to

$$L(x(t), \dot{x}(t)) = \frac{1}{2}m\dot{x}^2(t) - V(x(t)). \tag{3}$$

Next we introduce the so-called *action* [werking] *functional*

$$S[x] = \int_{t_1}^{t_2} L(x(t), \dot{x}(t))\, dt. \tag{4}$$

A functional maps a function, here $x(t)$, onto a number, here $S[x]$. The square brackets indicate that the argument is not a number but an entire function.

Hamilton's principle [principe van Hamilton] states that the time evolution of the system, $x(t)$, corresponds to a stationary point of the action, Eq. 4. More precisely, of all the curves $x(t)$ with given start point $x(t_1) = x_1$ and given end point $x(t_2) = x_2$ the true solution is a stationary point (either a minimum, maximum or saddle point) of the action.

We need now to define the meaning of a stationary point for a functional more precisely. We consider a small perturbation $h(t)$ around a given function $x(t)$. The new function $x(t)+h(t)$ needs to have the same start and end points, i.e., we require $h(t_1) = h(t_2) = 0$. Now let us consider

$$S[x+h] = \int_{t_1}^{t_2} L\left(x(t) + h(t), \dot{x}(t) + \dot{h}(t)\right) dt. \tag{5}$$

A Taylor expansion of the Lagrange function to first order leads to

$$S[x+h] = S[x] + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x} h + \frac{\partial L}{\partial \dot{x}} \dot{h}\right) dt + O\left(\|h\|^2\right) \tag{6}$$

where $O(\|h\|^2)$ stands for higher order terms, namely integrals that contain terms like $h^2(t)$ and $\dot{h}^2(t)$. Through integration by parts, namely replacing $\dot{h}\partial L/\partial \dot{x}$ by $\frac{d}{dt}(h\partial L/\partial \dot{x}) - h\frac{d}{dt}(\partial L/\partial \dot{x})$ and using the fact that the boundary terms vanish, one arrives at

$$S[x+h] - S[x] = \int_{t_2}^{t_2} \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right) h\, dt + O\left(\|h\|^2\right). \tag{7}$$

One says that $x(t)$ is a *stationary point* of the functional S if the integral vanishes for any small h. This is the case if $x(t)$ fulfills the so-called *Euler-Lagrange equation* [Euler-Lagrange-vergelijking]

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0. \tag{8}$$

Let us take the Lagrange function from above, Eq. 3, as an example. By inserting it into the Euler-Lagrange equation, Eq. 8, we find the equation of motion, Eq. 2. For this special case we can thus indeed verify that the time evolution of the system, the solution of Eq. 2, is a stationary point of the action, Eq. 4. It is straightforward to extend the formalism to d dimensions where one obtains d Euler-Lagrange equations, one for each direction in space. One can then easily verify that this set of equations is identical to the equations of motion for a particle in d dimensions.

So far it looks like Hamilton's principle is a very complicated way of obtaining the equation of motion, Eq. 2, that one can write down immediately. For more

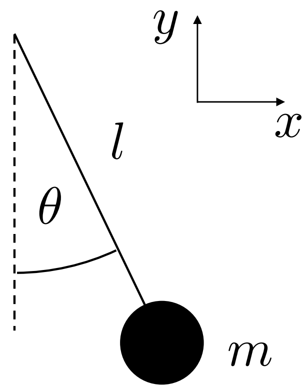


Figure 2: The pendulum.

complicated systems that contain certain constraints, however, such a framework is extremely useful. To give an example: Consider a pendulum, a mass m attached to a massless rod of length l that is suspended from a pivot at position $(x, y) = (0, 0)$ around which it can swing freely. The potential of the mass in the gravitational field is given by mgy. The Lagrange function of the pendulum is thus given by

$$L(x, y, \dot{x}, \dot{y}) = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) - mgy. \tag{9}$$

The Euler-Lagrange equations for the x- and y-coordinates lead to two equations of motion, $\ddot{x} = 0$ and $\ddot{y} = -g$.

Unfortunately these equations are completely wrong. What we found are the equations of motion of a free particle in 2 dimensions in a gravitational field. Solutions are e.g. trajectories of rain drops or of cannon balls but certainly not the motion of a pendulum. What went wrong? We forgot to take into account the presence of the rod that imposes a constraint, namely that $x^2 + y^2 = l^2$. A better approach would be to use a coordinate system that accounts automatically for this constraint, namely to describe the state of the pendulum by the angle $\theta(t)$ between the pendulum and the y-direction, see Fig. 2. But how does the equation of motion look in terms of this angle?

Here comes into play a great advantage of Hamilton's principle: it is independent of the coordinate system that one chooses. Suppose one goes from one coordinate system x_1, x_2,..., x_N to another coordinate system q_1, q_2,..., q_f via the transformations $\mathbf{q} = \mathbf{q}(\mathbf{x})$ and $\mathbf{x} = \mathbf{x}(\mathbf{q})$. The trajectory $\mathbf{x}(t)$ becomes then $\mathbf{q}(\mathbf{x}(t))$. The action functional can then be rewritten as

$$S[\mathbf{x}] = \int_{t_1}^{t_2} L(\mathbf{x}(t), \dot{\mathbf{x}}(t))\, dt = \int_{t_1}^{t_2} L\left(\mathbf{x}(\mathbf{q}(t)), \sum_{i=1}^{f} \frac{\partial \mathbf{x}(\mathbf{q}(t))}{\partial q_i}\dot{q}_i\right) dt. \tag{10}$$

The rhs of Eq. 10 is again of the form

$$S[\mathbf{q}] = \int_{t_1}^{t_2} \tilde{L}(\mathbf{q}(t), \dot{\mathbf{q}}(t))\, dt \tag{11}$$

with a new Lagrangian $\tilde{L}$. Also here Hamilton's principle must hold, i.e., the dynamic evolution of the system follows from the Euler-Lagrange equations

$$\frac{\partial \tilde{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{q}_i} = 0 \tag{12}$$

for $i = 1, ..., f$.

If we have a system with constraints we can sometimes introduce coordinates that automatically fulfill those constraints. The equations of motion are then simply given by the Euler-Lagrange equations in these coordinates. Let us go back to the pendulum. We describe now the configuration of the pendulum by the angle $\theta(t)$, see Fig. 2. In terms of this angle the kinetic energy of the pendulum is given by $ml^2\dot{\theta}^2/2$ and the potential energy by $-mlg\cos\theta$. This leads to the following Lagrange function:

$$L\left(\theta, \dot{\theta}\right) = \frac{ml^2}{2}\dot{\theta}^2 + mgl\cos\theta. \tag{13}$$

The corresponding Euler-Lagrange equation is given by

$$\ddot{\theta}(t) = -\frac{g}{l}\sin\theta(t), \tag{14}$$

which is indeed the equation of motion of the pendulum.

1.3 Generalized coordinates

In the previous example we have introduced so-called *generalized coordinates* [gegeneraliseerde coördinaten]. Generalized coordinates are any collection of independent coordinates q_i (independent means not connected by any equations of constraint) that are just sufficient to characterize the position of a system of particles.

In the previous case of a planar pendulum the pendulum body moves in the two-dimensional xy-plane. Its position is then given by (x, y). The system has, however, not two degrees of freedom but one. This is a consequence of the constraint $x^2 + y^2 = l^2$. $q_1 = x$ and $q_2 = y$ would thus not be an example of generalized coordinates but $q_1 = \theta$ is.

In general, if N particles are free to move in 3D but their $3N$ coordinates are related by m independent conditions of constraint, then the system has $f = 3N - m$ degrees of freedom and there are f independent generalized coordinates to describe them. Important is here that the constraints are expressible as equations of the form

$$f_j(x_1, x_2, x_3, ..., x_N, y_N, z_N, t) = 0 \quad \text{for} \quad j = 1, 2, ..., m. \tag{15}$$

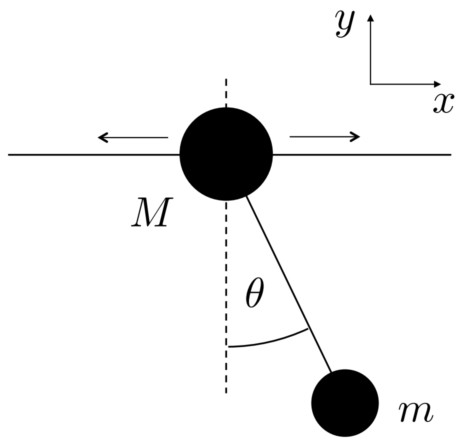


Figure 3: Example: a pendulum on a movable support.

Such constraint are called *holonomic*.

Remarkably the constraint for a cylinder rolling without slip on a surface is holonomic (i.e. the location of its centerline and its orientation are coupled through a holonomic constraint) but not for a sphere. For a very short rolling motion the orientation of the sphere and the location of its center is coupled (like for a cylinder). But through rolling of the sphere along suitable curves one can achieve that for every sphere position all possible orientations are possible. Such non-holonomic constraints are hard to deal with and will not be discussed here.

1.4 Examples of Lagrange equations

The best way to learn how Lagrange equations and generalized coordinates work is to look at specific examples.

Pendulum on a movable support Consider a mass M that can move freely along a horizonal line without friction. Attached to the mass M is a pendulum of mass m via a massless connection of length l (Fig. 3). We calculate now the Lagrange equations for this system.

We first need to find a suitable coordinate system. The system has 2 degrees of freedom (you can find this number by subtracting the two constraints from the 4 degrees of freedom of the unconstrained masses). Practical coordinates are the position X of the mass M along the line and the angle θ between the pendulum and the direction of gravity. The position of the pendulum body is then given by

$$x = X + l\sin\theta \quad \text{and} \quad z = -l\cos\theta.$$

The kinetic energy is then given by:

$$T = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\left(\dot{x}^2 + \dot{z}^2\right) = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\left[\left(\dot{X} + l\dot{\theta}\cos\theta\right)^2 + \left(l\dot{\theta}\sin\theta\right)^2\right].$$

This simplifies to

$$T = \frac{1}{2}\left(m + M\right)\dot{X}^2 + \frac{1}{2}m\left[2l\dot{X}\dot{\theta}\cos\theta + \left(l\dot{\theta}\right)^2\right].$$

The potential energy is given by

$$V = -mgl\cos\theta.$$

We can now obtain the equations of motions by taking derivatives of the Lagrangian with respect to the coordinates and to their time derivatives. This is done separately for the two coordinates. The Lagrange equation 12 for the coordinate X is given by:

$$\frac{d}{dt}\left[\frac{\partial\left(T - V\right)}{\partial\dot{X}}\right] - \frac{\partial\left(T - V\right)}{\partial X} = 0,$$

leading to

$$\left(m + M\right)\ddot{X} + ml\frac{d}{dt}\left(\dot{\theta}\cos\theta\right) = 0$$

or

$$\left(m + M\right)\ddot{X} = ml\left(\dot{\theta}^2\sin\theta - \ddot{\theta}\cos\theta\right).$$

Note that the partial derivative with respect to $\dot{X}$ is only taken on those places where this variable occurs but that the derivative with respect to the time t acts on all variables including θ en $\dot{\theta}$. Another point to note here is that quantity $\partial\left(T - V\right)/\partial\dot{X}$ is conserved (i.e. does not change with time). This follows always immediately if the Lagrangian does not depend on one of the coordinates (here X). You can check easily that this quantity is here the total momentum in the X-direction.

For the other coordinate, θ, we obtain:

$$\frac{d}{dt}\left[\frac{\partial\left(T - V\right)}{\partial\dot{\theta}}\right] - \frac{\partial\left(T - V\right)}{\partial\theta} = 0,$$

leading to

$$ml\left(l\ddot{\theta} + \ddot{X}\cos\theta - \dot{X}\dot{\theta}\sin\theta\right) + ml\dot{X}\dot{\theta}\sin\theta + mgl\sin\theta = 0$$

or

$$\ddot{\theta} + \frac{\ddot{X}}{l}\cos\theta + \frac{g}{l}\sin\theta = 0.$$

This example shows how straightforward the equations of motion can be derived with the Lagrange formalism as compared to deriving them from Newton's formalism which involves force vectors.

Particle in a central force field For a particle in a central field the motion takes place in a plane. We choose polar coordinates. The velocity has a radial and a tangential component. The kinetic and the potential energies are given by

$$T = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \quad \text{and} \quad V = V(r).$$

The Lagrange equation for r is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r} = \frac{\partial L}{\partial r} = mr\dot{\theta}^2 - \frac{dV}{dr}$$

and for θ:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m\frac{d}{dt}\left(r^2\dot{\theta}\right) = \frac{\partial L}{\partial \theta} = 0.$$

Here we find that $mr^2\dot{\theta}$, the *angular momentum* [impulsmoment, draaiimpuls, hoekmoment of draaimoment], is conserved as the Lagrangian does not depend on θ.

1.5 Conjugate momenta

We call the quantity

$$p_i = \partial L/\partial \dot{q}_i \tag{16}$$

the *conjugate momentum* [geconjugeerde impuls] to q_i. If the Lagrangian does not depend on a coordinate q_i (a so-called *ignorable coordinate*) we obtain from the corresponding Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0. \tag{17}$$

The conjugate momentum to q_i is then a constant of the motion.

As an example consider the motion of free particle in one dimension. The Lagrangian $L = T = m\dot{x}^2/2$ does not depend on x and thus $(d/dt)(\partial L/\partial \dot{x}) = 0$. This means $m\dot{x} = \text{const}$. Another example is the total momentum of an isolated system which is conjugated to the position of center of mass (see the following chapter for a definition of the center of mass). In the previous example (particle in a central force field) the Lagrangian does not depend on θ and the angular momentum $mr^2\dot{\theta}$ is a constant of motion.

1.6 Forces of constraint

The method of *Lagrange multipliers* [Lagrange multiplicatoren] is used in general if one wants to optimize (maximize or minimize a function) under one or several constraints. Suppose you want to maximize the function $f(x_1, ..., x_m)$. If this function has a maximum it must be one of the points where the function has zero slope, i.e., where its gradient vanishes: $\nabla f = 0$ with $\nabla = (\partial/\partial x_1, ..., \partial/\partial x_m)$. What do we have to do, however, if there is an additional

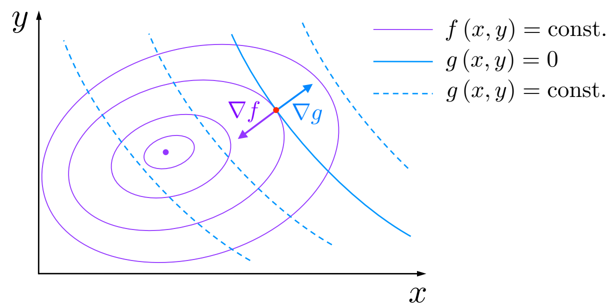


Figure 4: The method of the Lagrange multiplier. The objective is to find the maximum of the function $f\,(x_1, x_2)$ under the constraint $g\,(x_1, x_2) = C$. Shown are lines of equal height of f (purple curves) and of g (blue curves). The red point indicates the maximum of interest. It is the highest point of f on the line defined by $g = C$. At this point the gradients of the two height profiles are parallel or antiparallel (case shown here). This means there exists a number $\lambda \neq 0$, called the Lagrange multiplier, for which $\nabla f = \lambda \nabla g$.

constraint, $g\,(x_1, ..., x_m) = C$ with C some constant? This constraint defines an $(m-1)$-dimensional surface in the m-dimensional parameter space. Figure 4 explains the situation for $m = 2$. In that case $f\,(x_1, x_2)$ gives the height above (or below) the (x_1, x_2)-plane. As in a cartographic map we can draw contour lines for this function. The constraint $g\,(x_1, x_2) = C$ defines a single line g_C (or combinations thereof) in the landscape. The line g_C crosses contour lines of f. We are looking for the highest value of f on g_C. It is straightforward to convince oneself that this value occurs when g_C touches a contour line of f (if it crosses a contour line one can always find a contour line with a higher value of f that still crosses the g_C-line). Since g_C and the particular contour line of f touch tangentially, the gradients of the two functions at the touching point are parallel or antiparallel. In other words, at this point a number λ exists (positive or negative), called the Lagrange multiplier, for which

$$\nabla\,(f - \lambda g) = 0. \tag{18}$$

We use now the same procedure for the Lagrangian. We saw earlier that the Lagrange mechanics can deal easily with holonomic constraints. What we have found is then the equation of motion on the allowed manifold that is embedded inside the unconstrained configurational space. For instance, the pendulum is allowed to move on the surface of a sphere embedded in the three-dimensional space. What this method did not provide us with, however, is the force acting on the rod connecting the mass to the pivot point. This is the force that keeps the mass in the manifold of the allowed positions. Sometimes one would like to know the forces of constraint, e.g. an engineer who builds a bridge would like to know which forces the structural elements need to support during heavy traffic.

To get access to such forces of constraint we use a new Lagrangian. For simplicity, let us consider a system with one holonomic constraint $g(\mathbf{q}(t)) = 0$ with $\mathbf{q} = (q_1, q_2, ..., q_{f+1})$. Now consider the new Lagrangian

$$L'(\mathbf{q}(t), \dot{\mathbf{q}}(t), \lambda(t)) = L(\mathbf{q}(t), \dot{\mathbf{q}}(t)) + \lambda(t)\, g(\mathbf{q}(t)) \tag{19}$$

where we introduced the Lagrange multiplier λ as an additional variable of L'. This multiplier turns out to be a function of t, $\lambda = \lambda(t)$, just as the other variables are functions of t (the deeper reason being that we want to extremize a functional under a certain constraint and not just a function like in Fig. 4). Applying again Hamilton's variational principle (now for L' for small variations $\mathbf{q}(t)$ and $\lambda(t)$) one finds the following Euler-Lagrange equations:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \lambda(t)\frac{\partial g}{\partial q_i} = 0 \tag{20}$$

and

$$\frac{\partial L'}{\partial \lambda} - \frac{d}{dt}\frac{\partial L'}{\partial \dot{\lambda}} = 0. \tag{21}$$

The last equation is simple the constraint $g(\mathbf{q}(t)) = 0$. We have now $f+2$ equations that allow to determine $f+2$ unknown functions, $q_i(t)$ $(i = 1, ..., f+1)$ and $\lambda(t)$. This seems overly complicated as compared to the strategy discussed earlier where one finds f generalized coordinates and then writes down the f corresponding Euler-Lagrange equation. But through the introduction of the Lagrange multiplier we have gained something: the new quantity

$$Q_i = \lambda(t)\frac{\partial g}{\partial q_i} \tag{22}$$

is a *generalized force of constraint* [gegeneraliseerde bewegingbeperkende kracht] that acts on the system such that the constraint is always fulfilled.

How can one see that? Look at Eq. 20. Suppose we have a system of one particle in 3D in an external potential $V(\mathbf{x})$, then this can be rewritten as

$$\dot{\mathbf{p}} = -\nabla V(\mathbf{x}) + \lambda(t)\nabla g(\mathbf{x}).$$

On the lhs is the change in momentum, on the rhs are the forces, the first being the force from the external potential, the second the force of constraint that ensures that always $g(\mathbf{x}) = 0$. This force acts perpendicular to the surface on which the particle is allowed to move. For instance, for a pendulum one has $g(\mathbf{x}) = x^2 + y^2 + z^2 - l^2 = 0$ and $\nabla g(\mathbf{x})$ points indeed in the radial direction. The multiplier $\lambda(t)$ makes sure that the strength of the force, $|\lambda(t)\nabla g(\mathbf{x})|$, has always the right value to ensure the constraint.

If there are more than one constraint, one simply adds additional terms, each with its own Lagrange multiplier, to the Lagrangian. One finds then corresponding generalized forces of constraint. We note that these generalized forces are not always forces but can also be torques when the corresponding generalized coordinates are angular.

We give now a simple example for this method, the pendulum, see Fig. 2. We choose polar coordinates so that the position of the mass is given by $(x(t), y(t)) = (r(t)\sin\theta(t), r(t)\cos\theta(t))$. The constraint is $g(r) = r - l = 0$. The kinetic energy is

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

and the potential energy is $V = -mgr\cos\theta$. The new Lagrangian, Eq. 19, is now

$$L'\left(\theta, \dot{\theta}, r, \dot{r}, \lambda\right) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\cos\theta + \lambda(r - l).$$

We have now 3 Euler-Lagrange equations, one for θ (which turns out to be unimportant for our purpose), one for λ (which is just the constraint itself, see above) and one for r:

$$\frac{\partial L'}{\partial r} - \frac{d}{dt}\frac{\partial L'}{\partial \dot{r}} = mr\dot{\theta}^2 + mg\cos\theta + \lambda + m\ddot{r} = 0$$

from which follows (using $\ddot{r} = 0$):

$$\lambda = -mr\dot{\theta}^2 - mg\cos\theta.$$

The generalized force of constraint is

$$Q_r = \lambda\frac{\partial g}{\partial r} = \lambda = -mr\dot{\theta}^2 - mg\cos\theta.$$

This is the force that the massless rod has to sustain. Not surprisingly it is the sum of the centrifugal force and the radial component of the weight of the mass m.

1.7 Hamilton equations

Starting from Lagrange mechanics one can come to a different formulation by replacing the generalized velocities $\dot{q}_i$ by the conjugate momenta $p_i = \partial L/\partial q_i$ (Eq. 16). This leads to an alternative formulation of classical mechnics that is used for most advanced applications of theoretical mechanics and is used for the the transition from classical to quantum mechanics. To start we introduce the following function (still of q en $\dot{q}$):

$$H = \sum_i \dot{q}_i p_i - L. \tag{23}$$

One has

$$\sum_i \dot{q}_i p_i = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = \sum_i \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = 2T \tag{24}$$

because the kinetic energy is a homogeneous quadratic function of the $\dot{q}_i$'s (assuming $V = V(\mathbf{q})$). Specifically: with

$$T = \sum_{i,j} c_{ij}(q_k)\,\dot{q}_i\dot{q}_j \tag{25}$$

we find

$$\sum_i \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = \sum_i \dot{q}_i \left(2c_{ii}\dot{q}_i + \sum_{i \neq j} c_{ij}\dot{q}_j \right) = \sum_i 2c_{ii}\dot{q}_i^2 + 2\sum_{i \neq j} c_{ij}\dot{q}_i\dot{q}_j = 2T. \quad (26)$$

Hence

$$H = 2T - L = 2T - (T - V) = T + V. \quad (27)$$

The function H is thus the total energy of the system.

Instead of q_i, $\dot{q}_i$ we consider in the following q_i, p_i as the variables (position and momentum also play a symmetric role in quantum mechanics). This is possible because we can write $\dot{q}_i$ as a function of p_i en q_i: $\dot{q}_i = \dot{q}_i(p, q)$ (by solving Eq. 16 for $\dot{q}_i$). We find then the *Hamilton function*:

$$H(q, p) = \sum_i p_i \dot{q}_i(p, q) - L(q, \dot{q}(p, q)). \quad (28)$$

We calculate now the partial derivatives of H. We obtain

$$\frac{\partial H}{\partial p_j} = \dot{q}_j(q, p) + \sum_i \left(p_i \frac{\partial \dot{q}_i}{\partial p_j} - \frac{\partial L}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial p_j} \right) = \dot{q}_j \quad (29)$$

In the last step we used $p_i = \partial L / \partial \dot{q}_i$. Furthermore we obtain

$$\frac{\partial H}{\partial q_j} = \sum_i p_i \frac{\partial \dot{q}_i}{\partial q_j} - \frac{\partial L}{\partial q_j} - \sum_i \frac{\partial L}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial q_j} = -\frac{\partial L}{\partial q_j} \quad (30)$$

using again $p_i = \partial L / \partial \dot{q}_i$. From the Euler-Lagrange equation for q_j we then find

$$\frac{\partial H}{\partial q_j} = -\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} = -\dot{p}_j. \quad (31)$$

With this we have derived *Hamilton's canonical equations of motion* [Hamiltons canonische bewegingsvergelijkingen]:

$$\frac{\partial H}{\partial p_i}(q, p) = \dot{q}_i(t) \quad \text{and} \quad \frac{\partial H}{\partial q_i}(q, p) = -\dot{p}_i(t).$$

These are $2f$ coupled first-order differential equations instead of the f second-order differential equations of Lagrange.

As an example consider a harmonic oscillator in 1 dimension:

$$H(x, p) = T + V = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{p^2}{2m} + \frac{1}{2}kx^2.$$

Hamilton's equations are then given by

$$\frac{\partial H}{\partial p} = \frac{p}{m} = \dot{x}$$

and

$$\frac{\partial H}{\partial x} = kx = -\dot{p}.$$

These equations can be interpreted in an elegant way by introducing the *phase space* (faseruimte). This is the $2f$ dimensional space of all the positions and momenta of the particles. A point in this phase space corresponds to a particular state of the system. Hamilton's equations describe how the system races through phase space. For one particle in one dimension the phase space is two-dimensional. According to Hamilton's equations the trajectory of the harmonic oscillator would describe an ellipse in this phase space. The phase space of a system can have an incredibly high dimension. For instance, one mole of gas in a container (about 20 liters at atmospheric pressure) contains 6×10^{23} particles. Since each particle has 6 degrees of freedom, its phase space is 36×10^{23} dimensional. Despite of (or better because of) this high dimension one can easily deal with such system...but this is subject of another course, statistical mechanics.

2 Mechanics of a rigid body: planar motion

A *rigid body* [starre lichaam] is a system of particles where all particles have a fixed distance from each other. In this chapter we study rotations of a rigid body around a fixed axis, i.e., all particles move on planar circles. We relegate the more complex problem of the free motion of a rigid body to the next chapter.

2.1 Center of mass

Definition

The *center of mass* [zwaartepunt] gives the average position of a (not necessarily rigid) body. The averaging is done over the mass. For an isolated system of particles in an inertial system the center of mass moves always with constant velocity (constant speed and direction).

Discrete case: the body is made of masspoints with positions $\mathbf{r}_i$ and masses m_i. The center of mass is then given by the sum

$$\mathbf{r}_{\mathrm{CM}} = \frac{1}{M}\sum_i m_i \mathbf{r}_i \quad \text{with} \quad M = \sum_i m_i. \tag{32}$$

Continuous case: for a continuous distribution of masses the center of mass is given by the integral

$$\mathbf{r}_{\mathrm{CM}} = \frac{1}{M}\int \mathbf{r}\rho\left(\mathbf{r}\right) dV \quad \text{with} \quad M = \int \rho\left(\mathbf{r}\right) dV \tag{33}$$

Hier $\rho\left(\mathbf{r}\right)$ is the *density of mass* [massadichtheid] (mass per volume) and $dV = dxdydz$ is the volume element.

For a thin shell we can define the *area density* [oppervlaktedichtheid] $\sigma\left(\mathbf{r}\right)$ and the area element dS. The center of mass is now given by:

$$\mathbf{r}_{\mathrm{CM}} = \frac{1}{M}\int \mathbf{r}\sigma\left(\mathbf{r}\right) dS \quad \text{with} \quad M = \int \sigma\left(\mathbf{r}\right) dS. \tag{34}$$

For a thin wire with *line density* [lijndichtheid] $\lambda\left(\mathbf{r}\right)$ and length element dl we have

$$\mathbf{r}_{\mathrm{CM}} = \frac{1}{M}\int \mathbf{r}\lambda\left(\mathbf{r}\right) dl \quad \text{with} \quad M = \int \lambda\left(\mathbf{r}\right) dl. \tag{35}$$

Use of symmetry

If the distribution of mass has a symmetry then the center of mass must obey that symmetry as well. For instance, a body is mirrored onto itself by a reflection on the XY-plane. Every particle m_i with position (x_i, y_i, z_i) has mirror image m_i' at $(x_i', y_i', z_i') = (x_i, y_i, -z_i)$. As a result the center of mass lies on the symmetry plane:

$$z_{\mathrm{CM}} = \frac{1}{M}\sum_i \left(m_i z_i + m_i z_i'\right) = 0 \tag{36}$$

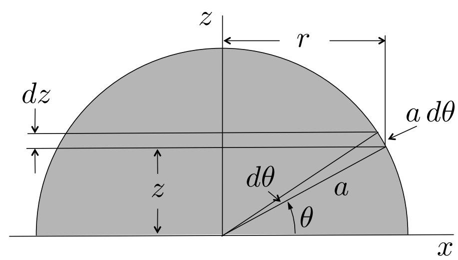


Figure 5: Coordinates used for calculating the center of mass of a solid hemisphere. With appropriate reinterpretation these coordinates are also used for a hemispherical shell, a semicircle and a half-disk.

Examples

- solid hemisphere with radius a and mass density ρ (see Fig. 5): Due to symmetry one has $x = y = 0$. What remains to be found is the height z_{CM} of the center of mass by integrating z over the hemisphere. For convenience we use disklike volume elements $dV = \pi r^2 dz$ and find

$$z_{\mathrm{CM}} = \frac{1}{2\pi a^3 \rho/3} \int_0^a z\pi \left(a^2 - z^2\right) \rho dz = \frac{3}{8}a. \tag{37}$$

- hemispherical shell: The surface element is given by $dS = 2\pi r \, a \, d\theta$ and its height by $z = a \sin\theta$ (see Fig. 5). This leads to

$$z_{\mathrm{CM}} = \frac{1}{2\pi a^2} \int_0^{\pi/2} a \sin\theta \, 2\pi a^2 \cos\theta d\theta = \frac{a}{2}. \tag{38}$$

- semicircle: $dl = ad\theta$ and

$$z_{\mathrm{CM}} = \frac{1}{\pi a} \int_0^{\pi} a \sin\theta \, a \, d\theta = \frac{2a}{\pi}. \tag{39}$$

- half-disk: Along similar lines one finds

$$z_{\mathrm{CM}} = \frac{4a}{3\pi}. \tag{40}$$

2.2 Rotation of a rigid body about a fixed axis

Each point of the body moves on a circle with *angular speed* [hoeksnelheid] ω (in rad/s) around the axis of rotation, say the z-axis. The speed of particle i

is $v_i = \omega r_i$ where $r_i = \sqrt{x_i^2 + y_i^2}$ denotes the distance of the particle from the axis of rotation. The velocity vector of the particle can be written as the cross product:

$$\mathbf{v}_i = \omega \mathbf{k} \times \mathbf{r}_i = \vec{\omega} \times \mathbf{r}_i. \tag{41}$$

Here $\mathbf{k}$ denotes the unit vector along the axis of rotation and $\vec{\omega} = \mathbf{k}\omega$ (the component of $\mathbf{r}_i$ along the z-axis, z_i, does not contribute to the velocity and indeed disappears in the cross product). The components of the cross product are

$$\dot{x}_i = -\omega y_i, \quad \dot{y}_i = \omega x_i, \quad \dot{z}_i = 0. \tag{42}$$

This is for the case of the rotation around the z-axis. Eq. 41 is also valid for rotations around an arbitrary axis.

Kinetic energy

The *kinetic energy* [kinetische energie] is given by

$$T_{\mathrm{rot}} = \sum_i \frac{1}{2} m_i v_i^2 = \frac{\omega^2}{2} \sum_i m_i r_i^2 = \frac{1}{2} I_z \omega^2 \tag{43}$$

with

$$I_z = \sum_i m_i \left(x_i^2 + y_i^2 \right). \tag{44}$$

I_z is called the *moment of inertia* [traagheidsmoment] about the z-axis.

Angular momentum

The *angular momentum* [impulsmoment] is given by

$$\mathbf{L} = \sum_i m_i \mathbf{r}_i \times \mathbf{v}_i. \tag{45}$$

Note that this vector is not necessarily pointing in the z-direction if not all points lie in the xy-plane. Let us calculate the z-component of the angular momentum. With Eqs. 42, 44 and 45 we obtain

$$L_z = \sum_i m_i \left(x_i \dot{y}_i - y_i \dot{x}_i \right) = \sum_i m_i \left(x_i^2 + y_i^2 \right) \omega = I_z \omega. \tag{46}$$

To see that $\mathbf{L}$ is not necessarily pointing along the axis of rotation, we calculate the vector triple product:

$$\mathbf{L} = \sum_i m_i \mathbf{r}_i \times (\vec{\omega} \times \mathbf{r}_i) = \sum_i m_i \left(r_i^2 \vec{\omega} - (\vec{\omega} \cdot \mathbf{r}_i) \mathbf{r}_i \right). \tag{47}$$

The first term points in the direction of rotation but the second not necessarily if $\vec{\omega} \cdot \mathbf{r}_i \neq 0$.

Equation of motion

In KMa it was shown that the rate of change of the angular momentum for any system is equal to the total *moment* [krachtmoment] exerted by the external forces:

$$\mathbf{N} = \sum_i \mathbf{r}_i \times \mathbf{f}_i = \sum_i m_i \mathbf{r}_i \times \dot{\mathbf{v}}_i = \frac{d}{dt} \sum_i m_i \mathbf{r}_i \times \mathbf{v}_i = \dot{\mathbf{L}}. \tag{48}$$

For the rotation around a fixed axis, e.g. the z-axis, one finds through projection on that axis:

$$N_z = \dot{L}_z = I_z \dot{\omega}. \tag{49}$$

Comparison translation - rotation

To summarize, the moment of inertia is for rotations what is the mass for translations. Specifically, for translations along the x-axis one has

$$p_x = m v_x, \quad T = \frac{1}{2} m v^2, \quad F_x = m \dot{v}_x$$

and for rotations about the z-axis one finds

$$L_z = I_z \omega, \quad T_{\mathrm{rot}} = \frac{1}{2} I_z \omega^2, \quad N_z = I_z \dot{\omega}.$$

2.3 Calculation of the moment of inertia

In the previous section we encountered the moment of inertia with respect to the z-axis, $I_z = \sum_i m_i \left(x_i^2 + y_i^2\right)$. For a continuous body this quantity is given by

$$I_z = \int r^2 dm \tag{50}$$

where dm denotes a mass element and r its distance from the axis of rotation. dm is given by the density factor multiplied by an appropriate differential: $dm = \rho(\mathbf{r})\, dV$ or $\sigma(\mathbf{r})\, dS$ or $\lambda(\mathbf{r})\, dl$. For composite bodies the total moment of inertia is the sum of the moments of the individual parts.

Examples

- thin rod of length L and mass $m = \lambda L$: If the axis of rotation is perpendicular to the rod and passes through its center we find

$$I_z = \int_{-L/2}^{L/2} x^2 \lambda dx = \frac{\lambda}{12} L^3 = \frac{m}{12} L^2. \tag{51}$$

- circular disk with radius a and mass $m = \pi a^2 \sigma$: We assume that the axis of rotation is perpendicular to the disk and passes through its center:

$$I_z = \int_0^a \sigma r^2 2\pi r dr = 2\pi \frac{\sigma a^4}{4} = \frac{m}{2} a^2. \tag{52}$$

- sphere of radius a and mass $m = 4\pi a^3 \rho/3$: We assume the axis to pass through the center and calculate the moment of inertia as an integral over disks. According to Eq. 52 the contribution of each disk with radius r is given by $r^2 dm/2$. Furthermore $dm = \rho\pi y^2 dz$. Hence

$$I_z = \int_{-a}^{a} \frac{1}{2}\pi\rho r^4 (z)\, dz = \int_0^{\pi/2} \frac{1}{2}\pi\rho \left(a^2 - z^2\right)^2 dz = \frac{8}{15}\pi\rho a^5 = \frac{2}{5} m a^2. \tag{53}$$

Perpendicular-axis theorem (or plane figure theorem)

Consider a rigid body that lies entirely in the z-plane, i.e., all mass points fulfill $z_i = 0$. Then

$$I_z = \sum_i m_i \left(x_i^2 + y_i^2\right) = \sum_i m_i x_i^2 + \sum_i m_i y_i^2 = I_x + I_y. \tag{54}$$

Example: We calculated above I_z of a circular disk, Eq. 51. What is I_x, i.e. the moment of inertia for a rotation about an axis that lies in the plane of the disk and passes through its center. From the perpendicular axis theorem follows immediately $I_x = I_y = ma^2/4$.

Parallel axis theorem (or Huygens-Steiner theorem)

Introduce coordinates $\bar{x}_i$ and $\bar{y}_i$ relative to the center of mass $(x_{\mathrm{CM}}, y_{\mathrm{CM}}, z_{\mathrm{CM}})$, i.e. $x_i = x_{\mathrm{CM}} + \bar{x}_i$ and $y_i = y_{\mathrm{CM}} + \bar{y}_i$. The moment of inertia with respect to the z-axis is given by

$$I_z = \sum_i m_i \left(x_i^2 + y_i^2\right) = \sum_i m_i \left(\left(x_{\mathrm{CM}} + \bar{x}_i\right)^2 + \left(y_{\mathrm{CM}} + \bar{y}_i\right)^2\right). \tag{55}$$

As the $\bar{x}_i$ and $\bar{y}_i$ are defined as the coordinates relative to the center of mass, we know that $\sum_i m_i \bar{x}_i = \sum_i m_i \bar{y}_i = 0$. This means that we can rewrite Eq. 55 as follows:

$$I_z = m\left(x_{\mathrm{CM}}^2 + y_{\mathrm{CM}}^2\right) + \sum_i m_i \left(\bar{x}_i^2 + \bar{y}_i^2\right) = m l_{\mathrm{CM}}^2 + I_{\mathrm{CM},z}. \tag{56}$$

The moment of inertia is the sum of two terms that reflects the fact that a rotation around the z-axis (which does not necessarily has to pass through the center of mass) is the superposition of the rotation of the body around the center of mass (second term in Eq. 56) and the rotation of the center of mass about the axis of rotation (first term). The angular momentum, Eq. 46, is then the sum of the angular momenta of these two contributions. From Eq. 56 we can see immediately that the moment of inertia is minimal if the axis of rotation goes through the center of mass.

2.4 The physical pendulum

A rigid body can swing freely around a fixed horizontal axis (say the y-axis) under its own weight. According to Eq. 49 the equation of motion is given by

$$I_y\ddot{\theta} = -mgl_{\mathrm{CM}}\sin\theta. \tag{57}$$

Here l_{CM} is the distance between the axis of rotation and the center of mass, θ gives the angular displacement. We used here the fact that the moment acting on the body is the same as if all the forces act on the center of mass. Note that this is true because of the linear relation between position and moment, Eq. 48; it does not hold for angular momentum because positions enter quadratically, Eq. 47.

For small angular displacements, $\theta \ll 1$ one can approximate $\sin\theta \approx \theta$ and we find the harmonic oscillator for which $\theta(t) = \theta_0\cos(2\pi f - \phi)$ with the frequency of oscillation:

$$f = \frac{1}{2\pi}\sqrt{\frac{mgl_{\mathrm{CM}}}{I_y}}. \tag{58}$$

The moment of inertia follows from parallel axis theorem, $I_y = I_{\mathrm{CM},y} + ml_{\mathrm{CM}}^2$. The period is thus given by

$$T = f^{-1} = 2\pi\sqrt{\frac{I_{\mathrm{CM},y}}{mgl_{\mathrm{CM}}} + \frac{l_{\mathrm{CM}}}{g}}. \tag{59}$$

Note that T goes to infinity for $l_{\mathrm{CM}} \to 0$ and for $l_{\mathrm{CM}} \to \infty$. In between it is minimal for a certain value of l_{CM}. From $\partial T/\partial l_{\mathrm{CM}}$ follows that this length is given by $l_{\mathrm{CM}} = \sqrt{I_{\mathrm{CM},y}/m}$. This quantity coincides with the so-called *radius of gyration* [gyratiestraal]. For a general body this is defined as the distance where a point with the same mass as the whole body has the same moment of inertia as for a rotation about an axis that passes through the center of mass.

2.5 A rigid body in planar motion

We extend now our analysis to cases where the rigid body does not only rotate around a fixed axis (like for the pendulum) but where the position of the axis (but not its orientation) is allowed to change as well. An example is a cylinder rolling down an inclined plane.

In the following we choose the point O' as the point around which we wish to calculate the motion of the rigid body. For a rolling cylinder this point could e.g. lie on the axis of rotation or on the contact line to the surface. The new positions and velocities of mass point i are related to the old ones via

$$\mathbf{r}_i = \mathbf{r}_0 + \mathbf{r}'_i, \quad \mathbf{v}_i = \mathbf{v}_0 + \mathbf{v}'_i. \tag{60}$$

Here $\mathbf{r}_i$ and $\mathbf{v}_i$ denote the positions and velocities of mass point i with respect to point of origin, O, of the intertial system and $\mathbf{r}_0 = \overrightarrow{OO'}$. We find for the moment with respect to point O':

$$\begin{aligned} \mathbf{N}' &= \sum_i \mathbf{r}'_i \times \mathbf{f}_i = \sum_i \mathbf{r}'_i \times \frac{d}{dt} m_i \left(\mathbf{v}_0 + \mathbf{v}'_i\right) \\ &= -\dot{\mathbf{v}}_0 \times \sum_i m_i \mathbf{r}'_i + \frac{d}{dt} \sum \mathbf{r}'_i \times m_i \mathbf{v}'_i. \end{aligned} \tag{61}$$

In the first term of the second line we gained a minus sign because we exchanged the order in the cross product, in the second term we moved the time derivative in front since $\dot{\mathbf{r}}'_i \times \mathbf{v}'_i \equiv \mathbf{v}'_i \times \mathbf{v}'_i = 0$. The second term gives just the time derivative of the angular momentum with respect to O':

$$\frac{d}{dt}\mathbf{L}' = \frac{d}{dt} \sum \mathbf{r}'_i \times m_i \mathbf{v}'_i. \tag{62}$$

Thus the equation of motion is modified to

$$\mathbf{N}' = -\dot{\mathbf{v}}_0 \times \sum_i m_i \mathbf{r}'_i + \frac{d}{dt}\mathbf{L}'. \tag{63}$$

Note that the usual relation between moment and angular momentum, Eq. 48, is still valid for a moving rigid body if its center of mass is chosen as the point O' (due to $\sum_i m_i \mathbf{r}'_i = 0$) or if the acceleration of O' vanishes.

To summarize, we can write down the following general equations of motion for the planar motion of a rigid body:

- for the translational motion: $\mathbf{F} = m\ddot{\mathbf{r}}_{\mathrm{CM}}$ where $\mathbf{F}$ is the vector sum of all external forces acting on the body,
- for the rotational motion with respect to some arbitrary origin O': $\mathbf{N}' = \dot{\mathbf{L}}' = I'\omega$, if the acceleration of O' vanishes or if O' passes through the center of mass.

Examples

- Cylinder rolling down an inclined plane without slip: Consider a cylinder of radius a with mass m. The forces on the cylinder are its weight, $m\mathbf{g}$, and the reaction of the surface with the components $\mathbf{F}_N$ and $\mathbf{F}_P$, see Fig. 6. For the component normal to the surface there is no acceleration and thus $0 = m\ddot{y} = mg\cos\theta - F_N$. The acceleration parallel to the surface

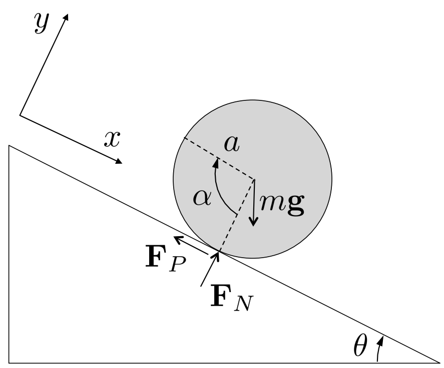


Figure 6: Cylinder rolling down an inclined plane.

follows from $m\ddot{x} = mg\sin\theta - F_P$. The momentum with respect to the center of mass (the torque exerted by the surface on the cylinder) is given by $N_{\text{CM}} = I\ddot{\alpha} = I\dot{\omega} = F_P a$. If the surface friction is large enough such that $\mu F_N > F_P$ (μ: coefficient of static friction) the surface can always apply enough torque on the cylinder to ensure that no slip occurs. In that case we have the holonomic constraint:

$$x = x_0 + a\alpha.$$

From this follows

$$m\ddot{x} = mg\sin\theta - F_P = mg\sin\theta - \frac{I}{a}\ddot{\alpha} = mg\sin\theta - \frac{I}{a^2}\ddot{x}$$

and thus

$$\ddot{x} = \frac{g\sin\theta}{1 + \frac{I}{ma^2}}.$$

For a cylinder one has obviously the same moment of inertia as for a disk, namely $I = ma^2/2$, 52, and thus

$$\ddot{x} = \frac{2}{3}g\sin\theta.$$

- Cylinder rolling down an inclined plane with slip: The surface can only provide the force not larger than $F_P = \mu F_N$. This value might be too small to enforce non-slippage. In such a case one has

  $$\ddot{x} = g\left(\sin\theta - \mu\cos\theta\right) \quad \text{and} \quad \ddot{\alpha} = \frac{mga}{I}\mu\cos\theta$$

  As an example consider a cylinder that is released a $t = 0$. The cylinder will slideif the acceleration of a point on the surface of the cylinder, $a\ddot{\alpha}$, is smaller than the acceleration of the contact point between cylinder and surface, $\ddot{x}$, i.e. if

$$\frac{mga^2}{I}\mu\cos\theta < g\left(\sin\theta - \mu\cos\theta\right).$$

There is a critical coefficient of friction given by

$$\mu_c = \frac{\tan\theta}{1+\frac{ma^2}{I}},$$

beyond which the cylinder rolls without friction.

3 Motion of a rigid body in three dimensions

In this chapter we consider the general motion of a rigid body where the direction of the rotational axis may vary.

3.1 Rotation of a rigid body around an arbitrary axis and a fixed point

We keep the laboratory system fixed and allow the rotation of a rigid body around an arbitrary axis that passes through the origin O and is allowed to change its direction freely. Suppose the system performs a rotation described the vector $\vec{\omega}$. According to Eq. 41 the rotational velocity of particle i of the rigid body is given by $\mathbf{v}_i = \vec{\omega} \times \mathbf{r}_i$. The angular momentum, Eq. 45, of the rigid body is given by:

$$\mathbf{L} = \sum_i m_i \mathbf{r}_i \times (\vec{\omega} \times \mathbf{r}_i) = \sum_i m_i \left[r_i^2 \vec{\omega} - (\vec{\omega} \cdot \mathbf{r}_i) \, \mathbf{r}_i \right] \tag{64}$$

where we used the expression of the vector triple product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c}) \, \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \, \mathbf{c}$. The relation between $\mathbf{L}$ en $\vec{\omega}$ is a linear function (mathematics) or a *tensor* [tensor] (physics). It is called *moment of inertia tensor* [traagheidstensor] $\mathbf{I}$. In mathematics we would write: $\mathbf{I} : \vec{\omega} \to \mathbf{L}$ but here we write simply

$$\mathbf{L} = \mathbf{I} \vec{\omega}. \tag{65}$$

The moment of inertia tensor can be written as a 3×3-matrix

$$\mathbf{I} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}. \tag{66}$$

Its components follow directly by writing out the components of the angular momentum, Eq. 64. For instance, the x-component is given by

$$L_x = \omega_x \sum_i m_i \left[x_i^2 + y_i^2 + z_i^2 \right] - \omega_x \sum_i m_i x_i^2 - \omega_y \sum_i m_i x_i y_i - \omega_z \sum_i m_i x_i z_i. \tag{67}$$

Thus from $L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$ it follows immediately that

$$I_{xx} = \sum_i m_i \left(y_i^2 + z_i^2 \right) \tag{68}$$

and

$$I_{xy} = -\sum_i m_i x_i y_i \tag{69}$$

and a corresponding relation for I_{xz}. Equivalent relation are found for all the other components. The diagonal elements of the matrix, I_{xx}, I_{yy} and I_{zz}, are simple the usual moments of inertia about the x-, y- and z-axes. New for us are the non-diagonal elements $I_{xy} = I_{yx}$, $I_{yz} = I_{zy}$ and $I_{zx} = I_{xz}$ which are called *products of inertia* [traagheidsproducten].

Examples We determine the moment of inertia tensor for a uniform square plate of size $l \times l$ and mass m with respect to its center of mass. The moment of inertia about an axis that lies in plane and is parallel to an edge is the same as for a rod, namely $ml^2/12$ (see 51). We can use the perpedicular axis theorem, Eq. 54, to determine the moment of inertia around an axis perpendicular to the the square, namely $ml^2/6$. The non-diagonal elements vanish, because $I_{xy} = -\int_{-l/2}^{l/2}\int_{-l/2}^{l/2} \sigma xy\, dxdy = 0$ due to symmetry and $I_{zx} = I_{zy} = 0$ since the plane lies in the $z = 0$ plane. In this coordinate system we find thus a diagonal moment of inertia tensor:

$$\mathbf{I} = \frac{1}{12}ml^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

We next determine the moment of inertia tensor of the same object but this time for rotations about axes that pass through the corner of the square. Two axes, x and y, coincide with the two associated edges of the plate, the third axis, z, is perpendicular to it. The diagonal elements $I_{xx} = I_{yy}$ we obtain through the parallel axis theorem, Eq. 56, namely $I_{xx} = ml^2/12 + ml^2/4$, the third, I_{zz}, has according to the perpendicular axis theorem twice that value. Again one has $I_{zx} = I_{zy} = 0$ since the plate lies in the $z = 0$-plane. But this time $I_{xy} = I_{yx}$ does not vanish:

$$I_{xy} = -\int_0^l \int_0^l \sigma xy dxdy = -\frac{\sigma l^4}{4} = -\frac{ml^2}{4}$$

Alltogether the moment of inertia tensor is given by:

$$\mathbf{I} = \frac{1}{12}ml^2 \begin{pmatrix} 4 & -3 & 0 \\ -3 & 4 & 0 \\ 0 & 0 & 8 \end{pmatrix}.$$

Kinetic energy The kinetic energy of a rotating rigid body is given by:

$$T_{\text{rot}} = \frac{1}{2}\sum_i m_i \mathbf{v}_i \cdot \mathbf{v}_i = \frac{1}{2}\sum_i m_i \mathbf{v}_i \cdot (\vec{\omega} \times \mathbf{r}_i)\,. \tag{70}$$

We can permute the vectors of the mixed product, $\mathbf{v}_i \cdot (\vec{\omega} \times \mathbf{r}_i) = \vec{\omega} \cdot (\mathbf{r}_i \times \mathbf{v}_i)$ and then put the vector $\vec{\omega}$ in front of sum. The latter step is allowed because we consider a rigid body. Using Eq. 45 we find:

$$T_{\text{rot}} = \frac{1}{2}\vec{\omega} \cdot \mathbf{L}. \tag{71}$$

Using Eq. 65 we obtain

$$T_{\text{rot}} = \frac{1}{2}\vec{\omega}\,\mathbf{I}\,\vec{\omega} \tag{72}$$

or in explicit matrix notation:

$$T_{\text{rot}} = \frac{1}{2} \begin{pmatrix} \omega_x & \omega_y & \omega_z \end{pmatrix} \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} . \tag{73}$$

For a rotation about a fixed axis we recover the result from the previous chapter. E.g. for a rotation around the Z-axis $\omega_x = \omega_y = 0$ and $\omega_z = \omega$ we obtain $T_{\text{rot}} = I_z \omega^2/2$ with $I_z = I_{zz}$.

3.2 Principal axes of a rigid body

In the matrix representation is becomes clear that the angular momentum has not necessarily the same orientation as the angular velocity vector, i.e. it is possible that $\mathbf{L} = \mathbf{I}\vec{\omega} \nparallel \vec{\omega}$. For three orthogonal directions, however, they are parallel. Note that the matrix $\mathbf{I}$ is symmetric and it has thus an orthonormal system of eigenvectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ with corresponding real eigen values I_1, I_2 and I_3. The eigenvectors are the *principal axes* [hoofdtraagheidsassen] of the rigid body for a given point O (often but not necessarily the center of mass). The three eigenvalues of the moment of inertia tensor are called *principal moments* [hoofdtraagheidsmomenten]. The eigenvalue and -vectors follow from the diagonlization of the moment of inertia matrix. The principle moments are obviously always positive. By aligning the coordinate system $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ with the principal axes of the body one has

$$\mathbf{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} . \tag{74}$$

Using this coordinate system expressions that describe the rotation around an arbitrary axis become fairly straightforward. Let $\mathbf{n}$ be the unit vector denoting the direction of the axis of rotation. Its components relative to the principle axes are given by direction cosines (see Fig. 7):

$$\mathbf{n} = \begin{pmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{pmatrix} \tag{75}$$

with $\cos^2\alpha + \cos^2\beta + \cos^2\gamma$ and $\vec{\omega} = \omega\mathbf{n}$. The moment of inertia about that axis is given by

$$I_{\mathbf{n}} = \mathbf{nIn} = I_1 \cos^2\alpha + I_2 \cos^2\beta + I_3 \cos^2\gamma \tag{76}$$

as we shall see in a moment.

Consider the angular momentum

$$\mathbf{L} = \mathbf{I}\omega\mathbf{n} = \omega \left(I_1 \cos\alpha\, \mathbf{e}_1 + I_2 \cos\beta\, \mathbf{e}_2 + I_3 \cos\gamma\, \mathbf{e}_3 \right) . \tag{77}$$

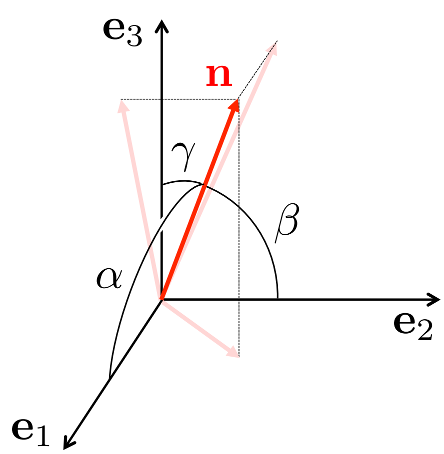


Figure 7: Definition of the direction cosines.

That quantity is not necessarily parallel to the rotation axis but its component in the direction of that axis is given by

$$\mathbf{L} \cdot \mathbf{n} = \omega \left(I_1 \cos^2 \alpha + I_2 \cos^2 \beta + I_3 \cos^2 \gamma \right) = I_{\mathbf{n}} \omega. \tag{78}$$

Finally the kinetic energy of the rotation obeys

$$T_{\rm rot} = \frac{1}{2} \vec{\omega}\, \mathbf{I}\, \vec{\omega} = \frac{1}{2} \omega^2 \left(I_1 \cos^2 \alpha + I_2 \cos^2 \beta + I_3 \cos^2 \gamma \right) = \frac{1}{2} I_{\mathbf{n}} \omega^2. \tag{79}$$

The surface of constant kinetic energy has the shape of an ellipsoid, the so-called the *inertia ellipsoid* [traagheidsellipsoïde]. When one choses the center of mass as the reference point, then this ellipsoid has a symmetry that cannot be smaller than that of the body. That simplifies the determination of the principal axes.

Determination of the other principal axes if one is known One of the principal axes might be known due to symmetry of the rigid body. If one choses that axes in the third direction and the other two axes arbitrarily, then the moment of inertia tensor is of the following form:

$$\mathbf{I} = \begin{pmatrix} I_{xx} & I_{xy} & 0 \\ I_{xy} & I_{yy} & 0 \\ 0 & 0 & I_3 \end{pmatrix}. \tag{80}$$

One can now diagonalize this matrix through a rotation in the xy-plane. This can be done by matrix multiplication by a rotation matrix $\mathbf{P}$:

$$\mathbf{I}' = \mathbf{P}^{-1} \mathbf{I} \mathbf{P} \quad \text{with} \quad \mathbf{P} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{81}$$

For simplicity we write only the $x-$ and $y-$components since the third component stays unchanged. We need to find a rotational angle θ such that:

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} I_{xx} & I_{xy} \\ I_{xy} & I_{yy} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} A & B \\ B & C \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix}$$

with $B = -(I_{xx} - I_{yy})\cos\theta\sin\theta + I_{xy}\left(\cos^2\theta - \sin^2\theta\right)$. In order to obtain a diagonal matrix we need $B = 0$ which is the case if

$$\frac{I_{xy}}{I_{xx} - I_{yy}} = \frac{\cos\theta\sin\theta}{\cos^2\theta - \sin^2\theta}. \tag{82}$$

With $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$ we obtain the following condition for the angle

$$\tan 2\theta = \frac{2I_{xy}}{I_{xx} - I_{yy}}. \tag{83}$$

Static and dynamic balancing Suppose a rigid body rotates around a fixed rotational axis (e.g. a car wheel). The rotation is called *statically balanced* if the axis of rotation lies on the center of mass. There can, however, still be a torque on the rotational axis, namely when it is not a principal axis. If the rotation is about a principal axis (that in addition goes through the center of mass) the device is *dynamically balanced*.

3.3 Euler's equation of motion of a rigid body

We finally are ready to derive the equation of motion of a rigid body under the action of external forces. The rotational part of the motion of any system referred to an inertial system is given by the equation of motion, Eq. 48: $\mathbf{N} = d\mathbf{L}/dt$. But $\mathbf{L}$ can only be expressed as a simple function of $\vec{\omega}$ in a coordinate system where the axes coincide with the principal axes of the body. In other words the coordinate system has to be fixed to the body and rotate with it. In KMa the theory of rotating coordinate systems has been developed. It has been shown that the time rate of change of a vector $\mathbf{V}$ in a fixed inertial system versus a rotating system (here the one attached to the rigid body) is given by (see Eq. (5.2.10a) in Fowles en Cassiday):

$$\left.\frac{d}{dt}\mathbf{V}\right|_{\text{fixed}} = \left.\frac{d}{dt}\mathbf{V}\right|_{\text{rot}} + \vec{\omega} \times \mathbf{V}. \tag{84}$$

The rate of change of the vector $\mathbf{V}$ in the fixed system is thus the sum of two terms, the rate of change of $\mathbf{V}$ with respect to the rotating system and the rate of change due to the rotation. This is also true for the angular momentum:

$$\left.\frac{d}{dt}\mathbf{L}\right|_{\text{fixed}} = \left.\frac{d}{dt}\mathbf{L}\right|_{\text{rot}} + \vec{\omega} \times \mathbf{L}. \tag{85}$$

Taking the time derivative of $\mathbf{L} = \mathbf{I}\,\vec{\omega}$ in the rigid body system we obtain

$$\frac{d}{dt}\mathbf{L}\bigg|_{\text{rot}} = \mathbf{I}\,\frac{d}{dt}\vec{\omega}\bigg|_{\text{rot}}$$

since $\mathbf{I}$ does not depend on time as the coordinate system rotates with the body. We thus find for the equation of motion *in the rotating system* (using Eq. 48):

$$\mathbf{N} = \mathbf{I}\frac{d}{dt}\vec{\omega} + \vec{\omega} \times \mathbf{L}. \tag{86}$$

To find the three components of this equation let us write explcitely

$$\vec{\omega} \times \mathbf{L} = \vec{\omega} \times \mathbf{I}\,\vec{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \times \begin{pmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{pmatrix} = \begin{pmatrix} \omega_2\omega_3\,(I_3 - I_2) \\ \omega_3\omega_1\,(I_1 - I_3) \\ \omega_1\omega_2\,(I_2 - I_1) \end{pmatrix}.$$

Therefore the three components of Eq. 86, the *Euler's equation of motion* [Eulervergelijkingen], are given by:

$$N_1 = I_1\dot{\omega}_1 + \omega_2\omega_3\,(I_3 - I_2)$$

$$N_2 = I_2\dot{\omega}_2 + \omega_3\omega_1\,(I_1 - I_3)$$

$$N_3 = I_3\dot{\omega}_3 + \omega_1\omega_2\,(I_2 - I_1)\,. \tag{87}$$

As an example consider the rotation of a rigid body with constant angular velocity about a fixed axis (that passes through the center of mass). Since the rotation vector is constant one has $N_1 = \omega_2\omega_3\,(I_3 - I_2)$ and two other similar equations for the other components. These are the components of the torque that need to be exerted to keep the axis of rotation fixed. For a rotation around a principal axis (e.g. the 1-axis) the torque vanishes because $\omega_2 = \omega_3 = 0$.

3.4 Free rotation: qualitative description

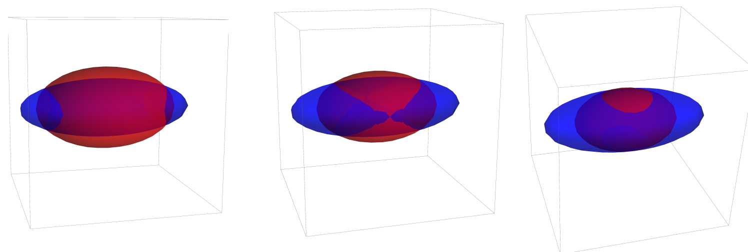
For a free rotation no external moment is exerted on the rigid body. According to Eq. 48 the angular moment stays constant in the fixed inertial system. The coordinate system attached to the rigid body rotates around $\mathbf{L}$. Since a rotation changes only the direction but not the length of $\mathbf{L}$, the following is a constant of the motion:

$$(I_1\omega_1)^2 + (I_2\omega_2)^2 + (I_3\omega_3)^2 = L^2. \tag{88}$$

Even though the components of $\vec{\omega}$ can vary, the tip of the $\vec{\omega}$-vector stays on an ellipsoid given by the relation above. Also the kinetic energy needs to stay constant since no external force is exerted on the body:

$$I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 = 2T_{\text{rot}}. \tag{89}$$

This second relation defines another ellipsoid with different ratios between the principal axes (it is more round).



(a) (b) (c)

Figure 8: Rigid body without rotational symmetry: 2 ellipsods, constant L (blue) and constant T_{rot} (red). T_{rot} decreases from (a) to (c). In case (a) and (c) the body moves along circles where the two elliposoids cut through each other. In case (b) the rotation is not stable.

Since the rotation vector needs to obey both relation, Eq. 88 and 89, the rotation vector needs lie on both surfaces. That means it needs to lie on the intersection of the two ellipsoids. InFig. 8 we show the two intersecting ellipsoids for a body without rotational symmetry (an asymmetric top, e.g. a book) with $I_1 > I_2 > I_3$. The blue ellipsoid corresponds to the surface of constant L, the red of constant T_{rot}. In the three examples we keep L constant but change the value of T_{rot}. In Fig. 8(a) we show the rotation around the 1-axis. If the rotation would go exactly around that principal axis, the two ellipsoids would touch other at that axis. For small variation, like in that figure, the angular velocity vector will move along a small ring around that axis. The same is true for the 3-axis, Fig. 8(c).

Very different is the case for the rotation about the 2-axis, corresponding to the intermediate principal moment I_2. The intersection between the two elliposids are curves that go all around the ellipsoids. The rotation around the 2-axis is thus not stable against small perturbation and the body will wildly change its orientation as soon as it is perturbed. This can be easily checked by throwing a book into the air inducing a rotation around any of the three principal directions.

3.5 Symmetric spinning top: free rotation

The Euler equation 87 for the free rotation of a symmetric spinning top, $N_1 = N_2 = N_3 = 0$, $I_3 = I_s$ and $I_1 = I_2 = I$, are given by

$$
\begin{aligned}
I\dot{\omega}_1 + \omega_2\omega_3\left(I_s - I\right) &= 0 \\
I\dot{\omega}_2 + \omega_3\omega_1\left(I - I_s\right) &= 0 \\
I_s\dot{\omega}_3 &= 0.
\end{aligned}
\tag{90}
$$

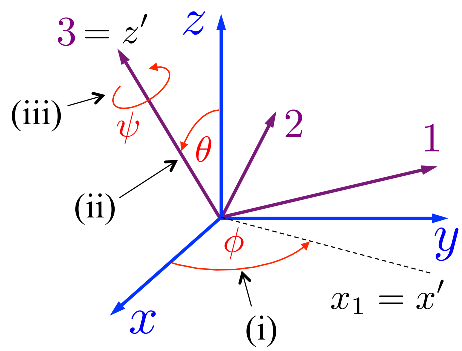


Figure 9: The three Euler angles give the orientation of the rotating coodinate system $(1, 2, 3)$ with respect to fixed system (x, y, z) of the observer.

It follows that ω_3 is a constant of motion. We define the reduced angular velocity

$$\Omega = \omega_3 \frac{I_s - I}{I} \tag{91}$$

allowing to rewrite the remaining two Euler equations:

$$\dot{\omega}_1 + \Omega\omega_2 = 0$$

$$\dot{\omega}_2 - \Omega\omega_1 = 0.$$

They can be combined to

$$\ddot{\omega}_1 + \Omega^2\omega_1 = 0. \tag{92}$$

This is the equation of the harmonic osciallator. Therefore the angular velocity vector describes a conical motion around the axis of symmetry with $\omega_1 = \omega_0 \cos\Omega t$, $\omega_2 = \omega_0 \sin\Omega t$ and $\omega_3 = \text{const}$. This motion is called *precession* [precessie]. The trajectory traced out by $\vec{\omega}$ is just the circle that we found in Fig. 8(a) (assuming $I_1 > I_2 = I_3$) as the intersection of the two ellipsoids.

3.6 Euler angles and the free rotation of a symmetric top

We have determined the motion of the angular velocity vector in the rotating coordinate system that is attached to the body. This gives us, however, not yet the motion of the body in the fixed coordinate system of the observer. To achive this, we need to describe the orientation of the rigid body with the help of angles, the *Euler angles* [Eulerhoeken] (Leonard Euler, 1776).

To go from the fixed coordinate system to that of the rotating body one needs three rotations by three angles, see Fig. 9:

(i) a rotation about the z-axis by the angle ϕ leads to new axes x_1 and y_1 ($z_1 \equiv z$),

(ii) a rotation around the x_1-axis by θ gives $x' \equiv x_1$, y' and z',

(iii) and a rotation around the z'-axis by ψ gives finally the body-fixed $(1, 2, 3)$-coordinate system.

Rotations do not commute so the order by which they are performed is important. But for infinitesimally small rotations can one simply add the variations of the positional vectors:

$$\vec{\omega}dt = d\vec{\omega} = d\phi\,\mathbf{e}_z + d\theta\,\mathbf{e}_{x'} + d\psi\,\mathbf{e}_{z'}. \tag{93}$$

It turns out that the (x', y', z')-coordinate system is most practical to discuss the motion of a symmetric top. The components of the angular velocity vector follow from Fig. 9 by inspection. Since $\mathbf{e}_z \cdot \mathbf{e}_{y'} = \sin\theta$ and $\mathbf{e}_z \cdot \mathbf{e}_{z'} = \cos\theta$ it follows from Eq. 93 that

$$\begin{aligned} \omega_{x'} &= \dot{\theta} \\ \omega_{y'} &= \dot{\phi}\sin\theta \\ \omega_{z'} &= \dot{\phi}\cos\theta + \dot{\psi}. \end{aligned} \tag{94}$$

For a free rotation the angular momentum vector $\mathbf{L}$ is a constant both with respect to its length and direction. We choose its direction as the z-axis. In the (x', y', z')-coordinates its components are given by:

$$\begin{aligned} L_{x'} &= 0 \\ L_{y'} &= L\sin\theta \\ L_{z'} &= L\cos\theta. \end{aligned} \tag{95}$$

The relation between the components of $\mathbf{L}$ and $\vec{\omega}$ follow from the moment of inertia tensor:

$$\begin{aligned} L_{x'} &= I\omega_{x'} \\ L_{y'} &= I\omega_{y'} \\ L_{z'} &= I_s\omega_{z'} \end{aligned} \tag{96}$$

where we used the symmetry of the tensor in the (x', y')-plane which is identical with the $(1, 2)$-plane.

Using these 9 equations, Eqs. 94, 95 and 96, we can now find the motion of the spinning top in the observer frame. First of all we obtain:

$$\omega_{x'} = 0,\ \dot{\theta} = 0. \tag{97}$$

This means that the angle θ is constant and that the angular velocity vector lies in the (y', z')-plane. We next determine the angle α between the angular velocity vector and the z'-axis. To do so we compare the y'- and z'-components of $\vec{\omega}$ and $\mathbf{L}$:

$$\begin{aligned} \omega_{y'} &= \omega\sin\alpha, \quad \omega_{z'} = \omega\cos\alpha \\ L_{y'} &= I\omega\sin\alpha, \quad L_{z'} = I_s\omega\cos\alpha. \end{aligned} \tag{98}$$

Thus

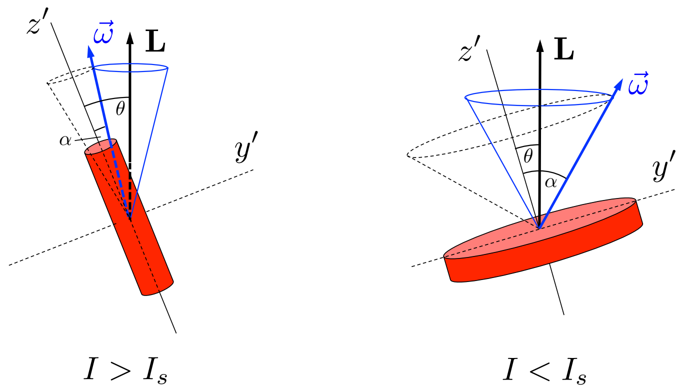


Figure 10: Free rotation of a symmetric top: elongated (lhs) and flat object (rhs).

$$\frac{L_{y'}}{L_{z'}} = \tan\theta = \frac{I}{I_s}\tan\alpha. \tag{99}$$

For $I > I_s$ (extended object, "cigar") we find that the angular velocity vector lies in between $\mathbf{L}$ and the symmetry axis (i.e. the z'-axis), $\theta > \alpha$, see lhs of Fig. 10. For a flat object, $I < I_s$, one has instead $\theta < \alpha$, see rhs Fig. 10.

The angular velocity of the x'-axis can be expressed as a function of θ alone or of α alone. For example, due to $\omega_{y'} = \dot{\phi}\sin\theta = \omega\sin\alpha$ one has

$$\dot{\phi} = \frac{\omega_{y'}}{\sin\theta} = \omega\frac{\sin\alpha}{\sin\theta} = \omega\sqrt{1 + \cos^2\alpha\left[\left(\frac{I_s}{I}\right)^2 - 1\right]}. \tag{100}$$

In the last step we used Eq. 99 and $\cos^2\theta = 1 - \sin^2\theta$ and the corresponding relation for α.

To summarize, there are three basic angular rates. (1) The angular velocity ω around the $\vec{\omega}$-axis. (2) The vector $\vec{\omega}$ and the symmetry axis (the 3- or z'-axis) rotate around the z-axis (the direction of $\mathbf{L}$) with the angular velocity $\dot{\phi}$, Eq. 100. (3) In the rotating $(1, 2, 3)$-system attached to the body the angular velocity vector rotates with the angular velocity Ω, Eq. 91, around the 3-axis, the symmetry axis of the body (and also $\mathbf{L}$ does this since $\mathbf{L}$ and $\vec{\omega}$ span a plane in which the 3-axis ($\equiv z'$-axis) lies, see Fig. 10).

3.7 Motion of a symmetric spinning top in the gravity field

In the following we study the motion of a symmetric spinning top under the action of a torque (caused e.g. by gravity) that is free to turn about a fixed point

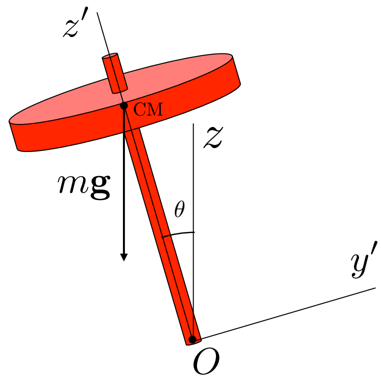


Figure 11: A symmetric spinning top under gravity.

O, see Fig. 11. We write down the Lagrangian in the(x', y', z')-coordinates:

$$L = \frac{1}{2}\left(I\omega_{x'}^2 + I\omega_{y'}^2 + I_s\omega_{z'}^2\right) - mgl\cos\theta$$

where $\omega_{x'}$, $\omega_{y'}$ and $\omega_{z'}$ are given by Eq. 94and l is the distance between point O and the center of mass. From this we find Euler-Lagrange equations (Eq. 8) for the three angles:

$$\frac{\partial L}{\partial \psi} = 0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\psi}} = I_s\frac{d}{dt}\left(\dot{\phi}\cos\theta + \dot{\psi}\right), \qquad (101)$$

$$\frac{\partial L}{\partial \phi} = 0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = \frac{d}{dt}\left[I\dot{\phi}\sin^2\theta + I_s\cos\theta\left(\dot{\phi}\cos\theta + \dot{\psi}\right)\right], \qquad (102)$$

$$\frac{dL}{d\theta} = mgl\sin\theta + I\dot{\phi}^2\sin\theta\cos\theta - I_s\dot{\phi}\sin\theta\left(\dot{\phi}\cos\theta + \dot{\psi}\right) = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\theta}}\right] = I\ddot{\theta}. \qquad (103)$$

From Eq. 101 we find:

$$\frac{d}{dt}\left(\dot{\phi}\cos\theta + \dot{\psi}\right) = 0 \qquad (104)$$

or $\dot{\phi}\cos\theta + \dot{\psi} = S$. S is a constant of motion, called *spin* [spin]. According to 94 one has $S = \omega_{z'}$, the component of the angular velocity about the symmtry axis. We thus find that $L_{z'} = I_sS$ (the conjugate momentum to ψ) is a constant of motion.

Inserting Eq. 104 into Eq. 102 one finds

$$\frac{d}{dt}\left[I\dot{\phi}\sin^2\theta + I_sS\cos\theta\right] = 0. \qquad (105)$$

We thus find a second constant of motion:

$$I\dot{\phi}\sin^2\theta + I_sS\cos\theta = L_z, \qquad (106)$$

the conjugate momentum of ϕ. That this is indeed the z-component of the angular momentum follows from $L_z = L_{y'} \sin\theta + L_{z'} \cos\theta$, see Fig. 11. It can be easily understood that L_z and $L_{z'}$ are constant since the torque points in the x'-direction and has thus no component along the z- and z'-direction.

Precession with a constant angle θ We look for a solution with $\dot{\theta} = 0$. From Eq. 103 we obtain in this case

$$I_s \dot{\phi} S \sin\theta - I \dot{\phi}^2 \sin\theta \cos\theta = mgl \sin\theta$$

or

$$I_s \dot{\phi} S - I \dot{\phi}^2 \cos\theta = mgl. \tag{107}$$

This is a second order equation for $\dot{\phi}$. It has only possible physical solutions for

$$I_s^2 S^2 \geq 4mglI \cos\theta. \tag{108}$$

This is the condition that needs to be met to have a stable precession with a constant θ. Especially, the spinning top can only stay in a vertical position (a so-called sleeping top) if

$$S > \frac{2\sqrt{mglI}}{I_s}. \tag{109}$$

Nutation: precession with a non-constant angle θ A precession with a constant angle θ requires a certain angular speed $\dot{\phi}$ of the precession, namely one of the two solutions of the second order equation 107. If the angular velocity of the precession is slower or faster, θ cannot stay constant. Instead the top performs a *nutation* [nutatie] between a minimal and a maximal value of θ on top of the precession as will be demonstrated in the lecture with the help of a gyroscope.

An example of a (nearly) symmetric spinning top is the precession of planet earth. The earth is not a free spinning top because of the tidal forces exerted by the sun and the moon. They cause the precession of the equinoxes, a cycle of approximately 26000 years. On top of this is a nutational motion with a very small opening angle of $0.3''$ (arcseconds) manifesting itself as a variation of the height of the poles. Its angular velocity follows from Eq. 91 with $\omega_3 = 2\pi/\text{day}$ and $(I_s - I)/I_s \approx 1/300$ due to the earth's oblateness. One predicts thus $\Omega \approx 300$ days, the actual value being quite close, about 418 days.