Beyond the Black-Scholes-Merton model

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Overview

Limitations of the Black-Scholes model

Black-Scholes model

Good news: it is a nice, well-behaved model

- simple, mathematically tractable
- no arbitrage, completeness
- explicit expressions for the basic options
- explicit hedging strategies
- can use PDE techniques for pricing
- can use stochastic analysis tools

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Bad news: it does *not* model what we actually see!...

- volatility smile
- statistical analysis

Assumptions in the Black-Scholes model

Asset prices modeled by a geometric Brownian motion:

$$S_t = S_0 e^{\mu t + \sigma W_t},$$

where

$$\mu: \text{ drift}, \quad \sigma: \text{ volatility}, \quad W: \text{ Brownian motion.}$$

Recall:

- $W_0 = 0$,
- $W_t - W_s$ is independent of $(W_u : u \leq s)$ for all $t \geq s$,
- $W_t - W_s \sim N(0, t-s)$,
- $t \mapsto W_t$ continuous.

Geometric Brownian motion

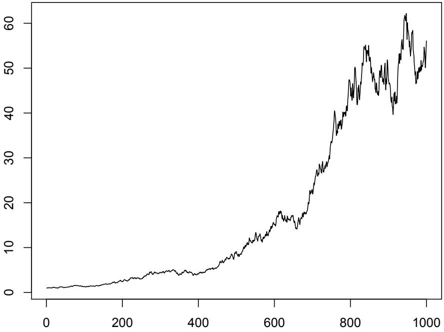

Figure: Typical sample path of a geometric Brownian motion

Returns in the Black-Scholes-Merton model

For $t \geq s$, let

$$R_{s,t} = \frac{S_t - S_s}{S_s}$$

be the return over the time interval $[s, t]$.

In the BSM-model:

$$R_{s,t} = e^{\mu(t-s)+\sigma(W_t - W_s)} - 1 \approx \mu(t-s) + \sigma(W_t - W_s).$$

Hence:

- returns over disjoint time intervals are independent,
- returns are (approximately) normally distributed.

Returns in the Black-Scholes-Merton model

Q: is this what we see in actual asset price data?

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Returns in the Black-Scholes-Merton model

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A: that depends...

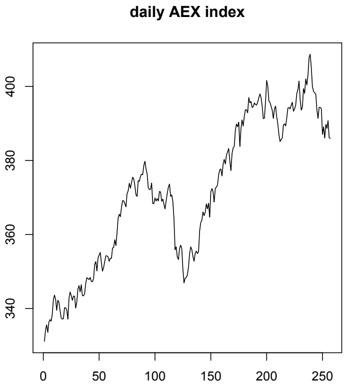
It turns out that the time-scale at which you look plays a crucial role!

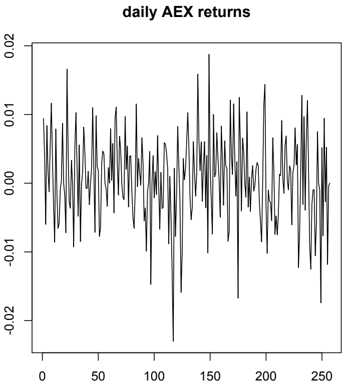
A look at some real asset price data

Data:

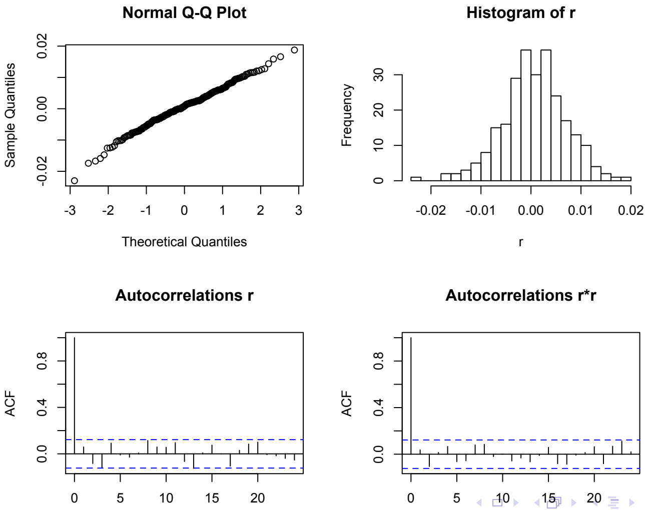
- Daily AEX index data
- Minute-by-minute Philips stockprice data

AEX data

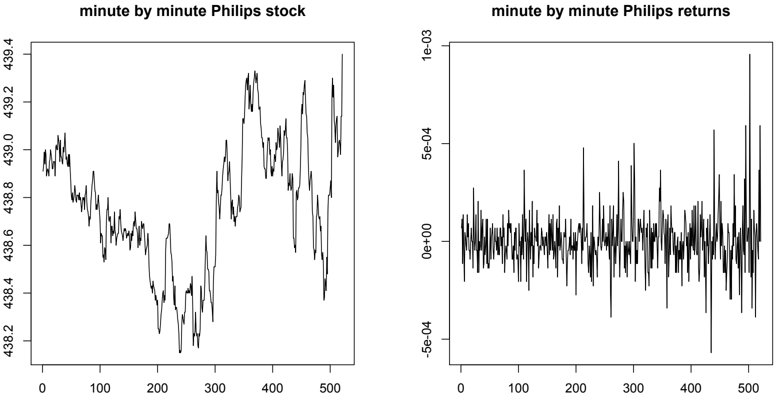

daily AEX index
400
380
360
340
0
50
100
150
200
250


daily AEX returns
0.02
0.01
0.00
-0.01
-0.02
0
50
100
150
200
250

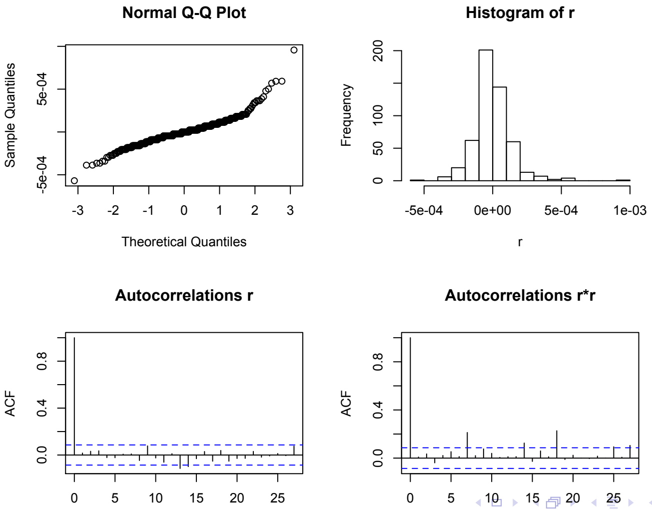
AEX data analysis



Philips data



Philips data analysis



Typical features of asset price data

- heavy-tailed returns
- squared returns are positively correlated
- long-range dependence in returns
- variable volatility
- volatility clustering

⋮

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Can we find sensible models that capture these features?

Issues surrounding model building for asset prices

What do we want?

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- Model should reflect observed statistical properties
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- Model should allow pricing/hedging
- Would like to have a "microscopic", "physical" justification

This turns out to be a lot to ask for...

Some attempts to construct better models

Stochastic volatility models

Idea: replace the constant volatility σ by a stochastic process.

Non-Brownian motion models

Idea: replace the Brownian motion by a different process driving the stock-price fluctuations.

Stochastic volatility models

Stochastic volatility

In the Black-Scholes-Merton model, the log-price process $X_t = \log S_t$ satisfies

$$dX_t = \mu\, dt + \sigma\, dW_t.$$

A stochastic volatility model postulates that

$$dX_t = \mu_t\, dt + \sigma_t\, dW_t,$$

for $(\sigma_t : t \geq 0)$ a stochastic process.

Why stochastic volatility?

- statistical properties: in general much better
- economic properties: incompleteness!
- pricing/hedging: involved/not always possible
- "microscopic", "physical" justification: perhaps reasonable?

Examples of stochastic volatility models

- Heston:

$$d\sigma_t^2 = \alpha(\theta - \sigma_t^2)\,dt + \tau\sqrt{\sigma_t^2}\,dB_t.$$

- GARCH-type:

$$d\sigma_t^2 = \alpha(\theta - \sigma_t^2)\,dt + \tau\sigma_t^2\,dB_t.$$

- 3/2 model:

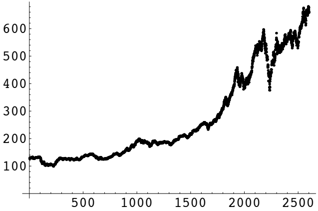
$$d\sigma_t^2 = \alpha(\theta - \sigma_t^2)\,dt + \tau(\sigma_t^2)^{3/2}\,dB_t.$$

$\vdots$

Which one should we use?

Nonparametric estimation of stochastic volatility models

Idea: use nonparametric estimator to choose the statistically most reasonable parametric model.



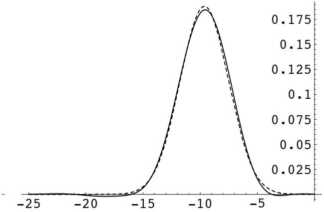


Figure: AEX data and estimator of the density of $\log \sigma_t^2$.

"Physical" justification

Reasonable that volatility has its own dynamics, partly independent of the dynamics of individual stocks ("temperature" of the market).

Volatility clustering and switching between "calm" and "excited" periods may be explained e.g. by a double well potential.

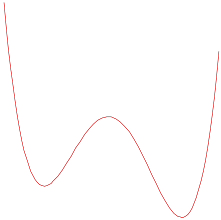


Figure: Potential V

SDE for volatility:

$$d\sigma_t^2 = V'(\sigma_t^2)\,dt + \tau(\sigma^2(X_t))\,dB_t,$$

where B is a second Brownian motion.

Completeness of the Black-Scholes market

The Black-Scholes market is complete: every "reasonable" contingent claim can be perfectly hedged by a self-financing portfolio consisting of stocks and bonds.

Intuitive reason: there are as many independent risky assets as sources of randomness. Or: by trading the stock the randomness caused by the Brownian motion can be "neutralized". In other words: there are enough risky assets to "hedge away" the randomness.

Technical reason: there is a unique martingale measure.

Incompleteness of stochastic volatility models

In SV-models typically more sources of randomness than risky assets.

As a result, not every derivative can be perfectly hedged in a SV model: incompleteness.

Some consequences:

- have to resort to e.g. super hedging or quantile hedging,
- for non-attainable derivatives, there is a whole interval of possible no-arbitrage prices,
- need additional considerations to choose a specific price (e.g. utility considerations).

Statistical soundness vs. completeness

Roughly speaking:

Models having the desirable property of completeness are typically not realistic from the statistical point of view.

Statistically sound models are typically incomplete and hence give rise to more involved pricing procedures.

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No widespread consensus...

Fractional Brownian motion models

Where does the Brownian motion come from?

Donsker ('52):

$Z_1, Z_2, \dots,$ i.i.d. $\mathbb{E}Z_i = 0$, $\mathbb{V}\mathrm{ar}Z_i = 1$.

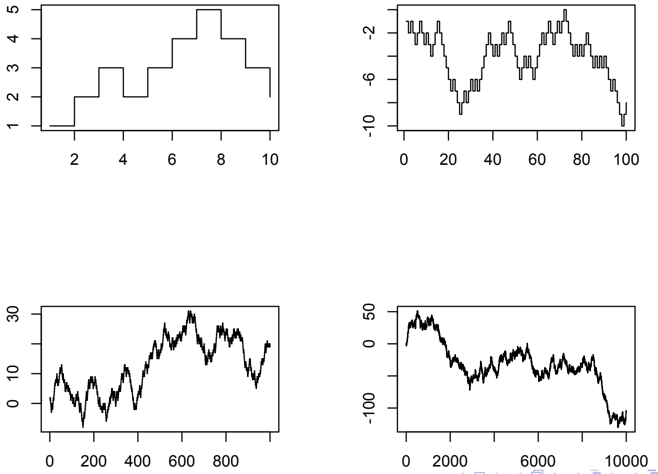
$$X_t^{(n)} = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} Z_i, \qquad t \in [0,1].$$

Theorem.

$$X^{(n)} \Rightarrow \text{Brownian motion}$$

in $D[0,1]$.

Donsker's theorem visualized



What if there is a memory in the system?

Davydov ('70), Taqqu ('75):

$Z_1, Z_2, \ldots,$ stationary, $\ldots,$ $\mathbb{E}Z_i = 0$, $\mathbb{V}\mathrm{ar}(Z_1 + \cdots + Z_n) \sim n^{2H}$, $H \in (0,1)$, $\ldots$

$$X_t^{(n)} = \frac{1}{\sqrt{\mathbb{V}\mathrm{ar}(Z_1 + \cdots + Z_n)}} \sum_{i=1}^{[nt]} Z_i, \qquad t \in [0,1].$$

Theorem.

$$X^{(n)} \Rightarrow \text{fractional Brownian motion}$$

in $D[0,1]$.

Fractional Brownian motion

Fractional Brownian motion (fBm):

Gaussian process $W^H = (W_t^H : t \geq 0)$, centered,

$$\mathbb{E} W_s^H W_t^H = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}),$$

with $H \in (0,1)$ the Hurst index.

Kolmogorov ('40), Mandelbrot & Van Ness ('68)

Fractional Brownian motion properties

Basic properties:

- $H = 1/2$: ordinary Brownian motion,
- stationary increments,
- H-self similar: for all $a > 0$, $(a^{-H} W^H_{at} : t \geq 0)$ has the same law as W^H,
- sample paths are "H-smooth",
- for $H > 1/2$, long range dependence: $\sum \mathbb{E}(W^H_n - W^H_{n-1})W^H_1 = \infty$,
- for $H \neq 1/2$: not Markov, not a (semi)martingale.
- $\mathbb{E}(W^H_t - W^H_s)^2 = |t - s|^{2H}$: for $H > 1/2$ the process is superdiffusive, for $H < 1/2$ it is subdiffusive.

Fractional Brownian motion paths

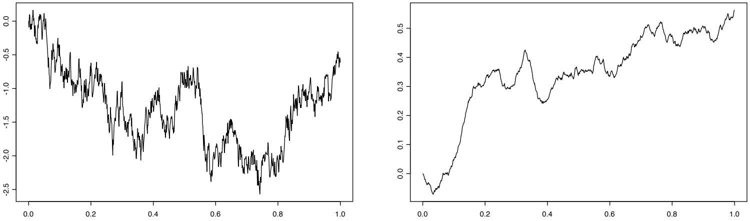


Figure: $H = 0.3$, $H = 0.8$

Fractional Black-Scholes model

Model for asset prices:

$$S_t = S_0 e^{\mu t + \sigma W_t^H},$$

where

μ : drift, σ : volatility, W^H : fractional Brownian motion.

Idea: Gaussianity plausible on appropriate time scales, fBm allows for more realistic dependence structure.

Data analysis studies: typically $H \approx 0.6$.

Properties of the fBm model

Good news:

- Statistical fit better than for BS model.
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Good news:

- Statistical fit better than for BS model.
- Microscopic explanation as scaling limit.

Bad news:

- Can not use stochastic calculus or PDE tools.
- The model allows for arbitrage!

Making sense of fBm models

Arbitrage opportunities arise in the usual setup:

- No transaction costs,
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Possible ways to remove arbitrage opportunities:

- Reduce the set of allowed trading strategies (e.g. only finitely many trading times).
- Introduce transaction costs (e.g. proportional to volumes traded).

Models with transaction costs

Recent developments:

After introducing transaction costs, BM can be replaced by any continuous process X with conditional full support:

$$\mathbb{P}\Big(\sup_{s\in[t,T]} |X_s - X_t - f(s)| < \varepsilon \,\Big|\, X_s : s \leq t\Big) > 0$$

for all $0 < t < T$, continuous $f : [t, T] \to \mathbb{R}$ with $f(0) = 0$, and $\varepsilon > 0$.

This leads to an arbitrage-free model.

Models with transaction costs

Let X be a Gaussian process with stationary increments and spectral measure $\mu(d\lambda) = f(\lambda)\, d\lambda$.

Theorem.

If

$$\int_1^\infty \frac{\log f(\lambda)}{\lambda^2}\, d\lambda > -\infty,$$

then X has conditional full support.

Example: for the fBm with Hurst index H, $f(\lambda) = c_H|\lambda|^{1-2H}$.
Hence, the fBm has CFS.

Models with transaction costs

For these models with general, non-semimartingale price processes:

- How about hedging?
- How about pricing?

Models with transaction costs

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- How about pricing?

Matters are currently unresolved. . .

Concluding remarks

- The Black-Scholes-Merton model does not properly describe all aspects of real asset price data.
- Stochastic volatility or fBm models typically do better.
- Stochastic volatility: incompleteness, how to choose derivative prices?
- Fractional Brownian motion: arbitrage, how to deal with it?
- The debate is ongoing. . .

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THANKS FOR YOUR ATTENTION!