EXAM QUANTUM THEORY January 18, 2007
(Course Dr. P.J.H. Denteneer, Fall 2006) 14.00 - 17.00h

Write NAME, INITIALS and STUDENT NUMBER on every sheet you hand in. Start each new problem on a new page.
All problems count for the same number of points (20) in the grading.

1) Consider a harmonic oscillator: $[\hat{a}, \hat{a}^\dagger] = \hat{1}$, $\hat{n} \equiv \hat{a}^\dagger \hat{a}$, $\hat{n}$-basis $\{|n\rangle\}$ with $n = 0, 1, 2, \ldots$, $\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$, $\hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$. An ensemble of such oscillators happens to be characterised by the following state (or: density) operator:

$$\hat{\rho} = \tfrac{1}{2}|2\rangle\langle 2| - \tfrac{i}{2}|2\rangle\langle 3| + \tfrac{i}{2}|3\rangle\langle 2| + \tfrac{1}{2}|3\rangle\langle 3|$$

(a) Determine the state (or: density) matrix in the n–representation, $\rho_{nn'}$.

(b) Does the given state operator describe a quantum system in a *pure state* or does it describe a *mixture*? Give an argument for your answer.

(c) In case the state operator is $\hat{\rho}$, determine the following expectation values (or: ensemble averages): $\langle a\rangle$, $\langle a^\dagger\rangle$ and $\langle n\rangle$.

2) Consider a spin $\frac{1}{2}$ object. Let the Hamilton operator be: $\hat{H} = -\hbar\omega\hat{\sigma}_y$. The normalised eigenstates of $\hat{\sigma}_y$ can be used as a basis and are given by:

$$|+, y\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + i\,|-\rangle\right) \quad \text{and} \quad |-, y\rangle = \frac{1}{\sqrt{2}}\left(i\,|+\rangle + |-\rangle\right) \ ,$$

in terms of the eigenstates $|+\rangle$ and $|-\rangle$ of $\hat{\sigma}_z$.

(a) Determine the matrix corresponding to $\hat{\sigma}_z$ in the σ_y-representation (i.e. in the basis of eigenstates of $\hat{\sigma}_y$).

(b) Argue that the (unitary) matrix U that transforms a general state vector in the σ_z-representation into the corresponding state vector in the σ_y-representation is given by:

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} .$$

Now, the state vector in the σ_z–representation at $t = 0$ is: $\chi(t = 0) = \begin{pmatrix} \cos\gamma \\ \sin\gamma \end{pmatrix}$.
Call the state vector in de σ_y–representation:

$$\eta(t) = \begin{pmatrix} e(t) \\ f(t) \end{pmatrix} .$$

(c) Determine $\eta(t = 0)$ and solve the Schrödinger-equation $i\hbar\frac{\partial}{\partial t}\eta(t) = \hat{H}\eta(t)$ in the σ_y–representation.

(d) Using the results derived in (a) and (c), show that the expectation value of $\hat{\sigma}_z$ at time t is given by:

$$\langle\sigma_z\rangle(t) = \cos\left(2\omega t - 2\gamma\right) .$$

3) Consider a single species of bosons with annihilation- and creation operators $\hat{a}$ and $\hat{a}^\dagger$, respectively. The Hamilton operator for this quantum many-body system is:

$$\hat{H} = \omega\left(\hat{a}^\dagger\hat{a} + \tfrac{1}{2}\right) + \tfrac{1}{2}\Delta\left(\hat{a}^\dagger\hat{a}^\dagger + \hat{a}\hat{a}\right) \ . \tag{1}$$

We take $\hbar = 1$ throughout this problem. The following transformation is useful to gain insight into the properties of this quantum system:

$$\hat{b} \ = \lambda\,\hat{a} + \mu\,\hat{a}^\dagger, \tag{2a}$$

$$\hat{b}^\dagger = \lambda^*\,\hat{a}^\dagger + \mu^*\,\hat{a}, \tag{2b}$$

where λ and μ are complex numbers.

(a) Show that the transformation (2) preserves the canonical commutation relations provided $|\lambda|^2 - |\mu|^2 = 1$.

(b) Assuming λ and μ to be real and using the result of (a), show that transformation (2) brings the Hamiltonian (1) into the form:

$$\hat{H} = \tilde{\omega}\left(\hat{b}^\dagger\hat{b} + \tfrac{1}{2}\right) \ . \tag{3}$$

Provide expressions for $\tilde{\omega}$, λ^2, and μ^2 in terms of ω and Δ.

(c) If the bosons characterized by $\hat{a}$ and $\hat{a}^\dagger$ are considered as excitations of a harmonic oscillator, the connection with the (Hermitian) operators for position and momentum of the oscillator, $\hat{x}$ and $\hat{p}$, is given by:

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\sqrt{m\omega}\ \hat{x} + \frac{i\,\hat{p}}{\sqrt{m\omega}}\right) \ . \tag{4}$$

Express the Hamiltonian in terms of $\hat{x}$ and $\hat{p}$ for the special case $\Delta = \omega$. How would you interpret this result physically?

4) Consider a many-body system corresponding to N-particle systems consisting of N identical spin-$\frac{1}{2}$ particles with Hamilton operator:

$$\hat{H}_N = \sum_{i=1}^{N} \hat{h}^{(i)} \quad , \quad \hat{h} = \frac{\hat{\vec{p}}^{\,2}}{2m} + B\,\hat{\sigma}_x \ .$$

Use the discrete $\vec{k}\sigma$–representation ($\hat{\vec{p}}|\vec{k}\sigma\rangle = \hbar\vec{k}|\vec{k}\sigma\rangle$, $\hat{\sigma}_z|\vec{k}\sigma\rangle = \sigma|\vec{k}\sigma\rangle$, $\sigma = +1, -1$).

(a) Give the fundamental algebraic relations of the annihilation- and creation-operators $\hat{a}_{\vec{k}\sigma}$ and $\hat{a}^\dagger_{\vec{k}\sigma}$.

(b) Show that the many-body energy operator $\hat{H}$ in the $\vec{k}\sigma$–representation is of the following form and determine f, g_+ and g_-:

$$\hat{H} = \sum_{\vec{k}\sigma} f\hat{a}^\dagger_{\vec{k}\sigma}\hat{a}_{\vec{k}\sigma} + \sum_{\vec{k}}\left(g_+\hat{a}^\dagger_{\vec{k},1}\hat{a}_{\vec{k},-1} + g_-\hat{a}^\dagger_{\vec{k},-1}\hat{a}_{\vec{k},1}\right) \ .$$

(c) Compute $\left[\hat{H}, \hat{a}_{\vec{k}\sigma}\right]$.

(d) Calculate the Heisenberg-picture operator $\hat{c}_{\vec{k}}(t)$ with $\hat{c}_{\vec{k}} \equiv \hat{a}_{\vec{k},1} + \hat{a}_{\vec{k},-1}$.

5) In classical mechanics a harmonic oscillator with mass m and (angular) frequency ω in one spatial dimension is given by the following Lagrangian L (the dot above x means differentiation with respect to time):

$$L(\dot{x}, x) = \tfrac{1}{2}m\dot{x}^2 - \tfrac{1}{2}m\omega^2 x^2 \quad .$$

The equation of motion follows from the Euler-Lagrange equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \quad .$$

The action $\mathcal{S}$ for the path $x(t)$ is given by:

$$\mathcal{S}[x(t)] = \int \mathrm{d}t \, L(\dot{x}, x) \quad ,$$

where the integral runs from begin- to end-time of the path.

(a) Calculate the action $\mathcal{S}_{\mathrm{cl}}$ for the classical path of the harmonic oscillator that starts at time $t = 0$ at position y and ends at time t' with *velocity* equal to: $-y\omega\sin(\omega t')$.

(b) The quantummechanical *propagator* for a harmonic oscillator that at time t_0 has position x_0 and at time t' has position x' is given by:

$$< x', t' | x_0, t_0 > = \langle x' | e^{-i\hat{H}(t'-t_0)/\hbar} | x_0 \rangle \quad ,$$

where the Hamiltonian is given by:

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + \tfrac{1}{2}m\omega^2\hat{x}^2 \ .$$

Show that the propagator for the harmonic oscillator with position x at $t = 0$ and position x' a *short* time interval Δt later is of the following form and determine the function $T(p, x, x')$:

$$\langle x' | e^{-i\hat{H}\Delta t/\hbar} | x \rangle = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} \mathrm{d}p \, \exp\left\{ iT(p, x, x')\Delta t/\hbar \right\} + \mathcal{O}\left(\Delta t\right)^2 \ .$$

If necessary, use that the scalar product of eigenstates of $\hat{x}$ en $\hat{p}_x$ is given by:

$$\langle x | p \rangle = \frac{1}{\sqrt{2\pi\hbar}} \, e^{ipx/\hbar} \ .$$