

The Domino Effect

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Abstract

The physics of a row of toppling dominoes is discussed. In particular the forces between the falling dominoes are analyzed and with this knowledge, the effect of friction has been incorporated. A set of limiting situations is discussed in detail, such as the limit of thin dominoes, which allows a full and explicit analytical solution. The propagation speed of the domino effect is calculated for various spatial separations. Also a formula is given, which gives explicitly the main dependence of the speed as function of the domino width, height and interspacing.

1 Introduction

Patterns formed by toppling dominoes are not only a spectacular view, but their dynamics is also a nice illustration of the mechanics of solid bodies. One can study the problem on different levels. Walker [1] gives a qualitative discussion. Banks [2] considers the row of toppling dominoes as a sequence of independent events: one domino undergoes a free fall, till it hits the next one, which then falls independently of the others, and so on. He assumes that in the collision the linear momentum along the supporting table is transmitted. This is a naive viewpoint, but it has the advantage that the calculation can be carried out analytically. A much more thorough treatment has been given by D. E. Shaw [3]. His aim is to show that the problem is a nice illustration of computer aided instruction in mechanics. He introduces the basic feature that the domino, after having struck the next one, keeps pushing on it. So the collision is completely *inelastic*. In this way a train develops of dominoes leaning on each other and pushing the head of the train. One may see this as an elementary demonstration of a propagating soliton, separating the fallen dominoes from the still upright ones. Indeed Shaw's treatment is a simple example how to handle holonomous constraints in a computer program describing the soliton. As collision law he takes conservation of angular momentum. We will demonstrate, by analyzing the forces between the dominoes, that this is not accurate. The correction has a substantial influence on the soliton speed, even more important than the inclusion of friction, which becomes possible when the forces between the dominoes are known.

The setting is a long row of identical and perfect dominoes of height h , thickness d and interspacing s . In order to make the problem tractable we assume that the dominoes only rotate (and e.g. do not slip on the supporting table). Their fall is due to

the gravitational force, with acceleration g . The combination \sqrt{gh} provides a velocity scale and it comes as a multiplicative factor in the soliton speed. Typical parameters of the problem are the aspect ratio d/h , which is determined by the type of dominoes used, and the ratio s/h , which can be easily varied in an experiment. Another characteristic of the dominoes is their mutual friction coefficient μ which is a small number (~ 0.2). The first domino gets a gentle push, such that it topples and makes a “free rotation” till it strikes the second. After the collision the two fall together till they strike the third and so forth. So we get a succession of rotations and collisions, the two processes being governed by different dynamical laws. Without friction the rotation conserves energy, while the constraints exclude the energy to be conserved in the collision. In fact this is the main dissipative element, more than the inclusion of friction.

The goal is to find the dependence of the soliton speed on the interdistance s/h . In the beginning this speed depends on the initial push, but after a while a stationary pattern develops: a propagating soliton with upright dominoes in front and toppled dominoes behind. The determination of the forces between the dominoes requires that we first briefly outline the analysis of Shaw. Then we analyze the forces between the dominoes. Knowing these we make the collision law more precise. With the proper rotation and collision laws we give the equations for the fully developed solitons. The next point is the introduction of friction and the calculation of its effect on the soliton speed. As illustration we discuss the limit of thin dominoes $d \rightarrow 0$, with permits for small interseparations a complete analytical solution. Finally we present our results for the asymptotic soliton speed for various values of the friction and compare them with some experiments. We also give an explicit formula, which displays the main dependence of the soliton speed on the parameters of the problem. The paper closes with a discussion of the results and the assumptions that we have made.

2 Constraints on the Motion

The basic observation is that domino i pushes over domino $i+1$ and *remains in contact afterwards*. So after the contact of i with $i+1$ the motion of i is constrained by the motion of $i+1$. Therefore we can take the tilt angle θ_n of the foremost falling domino, as the only independent mechanical variable (see Fig. 1). Simple goniometry tells that

$$h \sin(\theta_i - \theta_{i+1}) = (s + d) \cos \theta_{i+1} - d. \quad (1)$$

To see this relation it helps to displace domino $i+1$ parallel to itself, till its bottom line points at the rotation axis of domino i (see Fig. 1). By this relation one can express the tilt angle θ_i in terms of the next θ_{i+1} and so on, such that all preceding tilt angles are expressed in terms of θ_n . The recursion defines θ_i as a function of θ_n of the form

$$\theta_i = p_{n-i}(\theta_n), \quad (2)$$

i.e. the functional dependence on the angle of the head of the train depends only on the distance $n - i$. The functions $p_j(\theta)$ satisfy

$$p_j(\theta) = p_{j-1}(\theta) + \arcsin \left(\frac{(s + d) \cos p_{j-1}(\theta) - d}{h} \right), \quad (3)$$

with the starting function $p_0(\theta) = \theta$. They are defined on the interval $0 < \theta < \theta_c$, where θ_c is the angle of rotation at which the head of the train hits the next domino

$$\theta_c = \arcsin(s/h). \quad (4)$$

We will call θ_c the angular distance. From the picture it is clear that the functions are bounded by the value θ_∞ , which is the angle for which the right hand side of (1) vanishes

$$\cos \theta_\infty = \frac{d}{s+d}. \quad (5)$$

θ_∞ is the angle at which the dominoes are stacked against each other at the end of the train. We call θ_∞ the stacking angle.

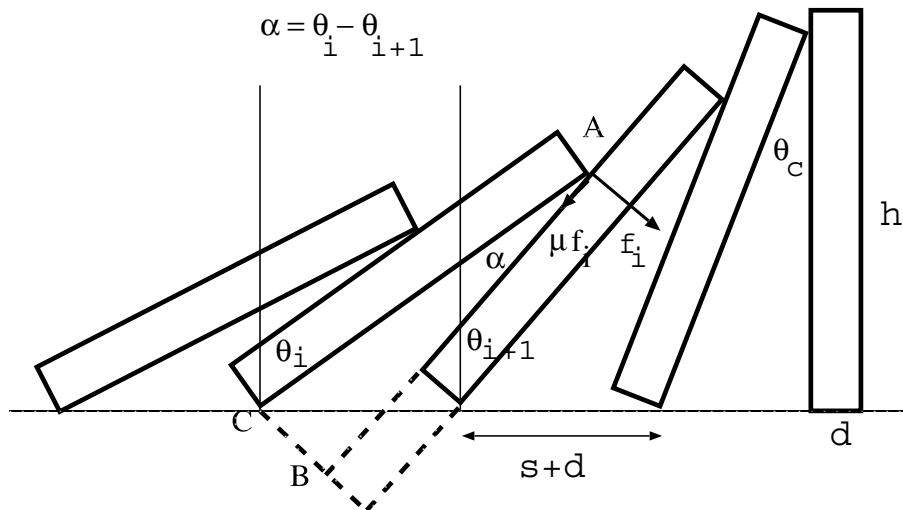


Figure 1: Successive dominoes. The tilt angle θ_i is taken with respect to the vertical. In the rectangular triangle ABC the top angle is $\alpha = \theta_i - \theta_{i+1}$, the hypotenuse has the length h and the base BC the length $(s+d) \cos \theta_{i+1} - d$. Expressing this base in the hypotenuse and the top angle yields relation (1). In the picture the tilt angle of the head of chain θ_n has reached its final value $\theta_c = \arcsin(s/h)$. The first domino has almost reached the stacking angle θ_∞ . The normal force f_i and the friction force μf_i that domino i exerts on $i+1$ are also indicated.

The picture shows that the functions $p_j(\theta)$ are monotonically increasing functions. They become flatter and flatter with the index j and converge to the value θ_∞ (at least not too close to the maximum separation $s = h$, see Section 10). The functions are strongly interrelated, not only by the defining recursion (3). The angle θ_i can be calculated from the head of the train θ_n by p_{n-i} but also from an arbitrary intermediate θ_k by p_{k-i} . This implies

$$p_{n-i}(\theta) = p_{k-i}(p_{n-k}(\theta)), \quad \text{e.g.} \quad p_j(\theta) = p_{j-1}(p_1(\theta)). \quad (6)$$

One easily sees that $p_1(0) = \theta_c$. Therefore one has

$$p_j(0) = p_{j-1}(p_1(0)) = p_{j-1}(\theta_c), \quad (7)$$

a property that will be used later on several times.

An immediate consequence of (1) is the expression for the angular velocities $\omega_i = d\theta/dt$ in terms of ω_n . From the chain rule of differentiation we find

$$\omega_i = \frac{d\theta_i}{d\theta_n} \frac{d\theta_n}{dt} = w_{n-i} \omega_n, \quad (8)$$

with

$$w_j(\theta) = \frac{dp_j(\theta)}{d\theta}. \quad (9)$$

Computationally it is easier to calculate the w_j recursively. Differentiation of (3) with respect to θ_n yields

$$w_j(\theta) = w_{j-1}(\theta) \left(1 - \frac{(s+d) \sin p_j(\theta)}{h \cos [p_j(\theta) - p_{j-1}(\theta)]} \right). \quad (10)$$

Another useful relation follows from differentiation of the second relation (6)

$$w_j(\theta) = w_{j-1}(p_1(\theta)) w_1(\theta) \quad \Rightarrow \quad w_j(0) = w_{j-1}(\theta_c), \quad (11)$$

since $p_1(0) = \theta_c$ and $w_1(0) = 1$.

3 Rotation Equations

Without friction, the motion between two collisions is governed by conservation of energy, which consists out of a potential and a kinetic part. The potential part derives from the combined height of the center of mass of the falling dominoes, for which we take the dimensionless quantity

$$H_n(\theta_n) = \sum_i^n [\cos \theta_i + (d/h) \sin \theta_i]. \quad (12)$$

The kinetic part is given by the rotational energy, for which holds

$$K_n(\theta_n, \omega_n) = (I/2) \sum_i^n \omega_i^2, \quad I = (1/3)m(h^2 + d^2), \quad (13)$$

where I is the angular moment of inertia with respect to the rotation axis and m is the mass of the dominoes. We write the total energy as

$$E_n = \frac{1}{2}mgh e_n = \frac{1}{2}mgh \left(H_n(\theta_n) + \frac{I}{mgh} I_n(\theta_n) \omega_n^2 \right), \quad (14)$$

where the dimensionless effective moment of inertia $I(\theta_n)$ is defined as

$$I_n(\theta_n) = \sum_j^n w_j^2(\theta_n). \quad (15)$$

We have factored out $mgh/2$ in (14) as it is an irrelevant energy scale. This has the advantage that the expression between brackets is dimensionless. The factor I/mgh

$$\frac{I}{mgh} = \frac{h(1 + d^2/h^2)}{3g} \quad (16)$$

provides a time scale that can be incorporated in ω_n . From now on we put this factor equal to unity in the formulae and remember its value when we convert dimensionless velocities to real velocities.

We see (14) as the defining expression for ω_n as function of θ_n

$$\omega_n(\theta_n) = \left(\frac{e_n - H_n(\theta_n)}{I_n(\theta_n)} \right)^{1/2}. \quad (17)$$

As mentioned e_n is a constant during interval n . So we can solve the temporal behavior of θ_n from the equation

$$\frac{d\theta_n(t)}{dt} = \omega_n(\theta_n). \quad (18)$$

The initial value for θ_n is 0 and the final value equals the rotational distance θ_c . The duration of the time interval where n is the head of the chain, follows by integration

$$t_n = \int_0^{\theta_c} \frac{d\theta_n}{\omega_n(\theta_n)}. \quad (19)$$

In this time interval the soliton has advanced a distance $s + d$. The ratio $(s + d)/t_n$ gives the soliton speed, when the head of the train is at n . In order to integrate the equations of motion (18) we must have a value for e_n which basically amounts to finding an initial value $\omega_n(0)$ as one sees from (14). In the next section we outline how to calculate successively the $\omega_n(0)$.

Putting all ingredients together we obtain the asymptotic soliton speed v_{as} as

$$v_{\text{as}} = \sqrt{gh} \left(\frac{3}{1 + d^2/h^2} \right)^{1/2} \frac{s + d}{h} \lim_{n \rightarrow \infty} \frac{1}{t_n}. \quad (20)$$

In this formula the time t_n is computed from the dimensionless equations (setting I/mgh equal to 1).

4 The Collision Law, first version

We now investigate what happens when domino n hits $n + 1$. In a very short time domino $n + 1$ accumulates an angular velocity $\omega_{n+1}(0)$. The change in ω_{n+1} takes place while the tilt angles of the falling dominoes hardly change. Shaw [3] postulates that the total angular momentum of the system is unchanged during the collision. This is not self-evident and we comment on it in Section 6. Before the collision we have the angular momentum

$$L_n = \sum_j^n w_j(\theta_c) \omega_n(\theta_c). \quad (21)$$

After the collision we have

$$L_{n+1} = \sum_j^{n+1} w_j(0) \omega_{n+1}(0). \quad (22)$$

Equating these two expressions yields the relation

$$\omega_{n+1}(0) = \omega_n(\theta_c) \frac{\sum_j^n w_j(\theta_c)}{\sum_j^{n+1} w_j(0)}. \quad (23)$$

With the aid of this value we compute the total energy e_{n+1} and the next integration can be started. For the first time interval holds $e_0 = 1 + \omega_0^2(0)$ since only the zeroth domino is involved and it starts in upright position with angular velocity $\omega_0(0)$. The value of $\omega_0(0)$ has no influence on the asymptotic behavior. After a sufficient number of time intervals, a stationary soliton develops.

5 Forces between the Dominoes

Conservation of energy requires the dominoes to slide frictionless over each other. Before we can introduce friction we have to take a closer look at the forces between the falling dominoes. Without friction the force which i exerts on $i + 1$ is perpendicular to the surface of $i + 1$ with a magnitude f_i (see Fig. 1). Consider to begin with the head of the train n . Domino n feels the gravitational pull with a torque T_n

$$T_n = (\sin \theta_n - (d/h) \cos \theta_n)/2, \quad (24)$$

and a torque from domino $n - 1$ equal to the force f_{n-1} times the moment arm with respect to the rotation point of n . The equation of motion for n becomes

$$\frac{d\omega_n}{dt} = T_n + f_{n-1} h [\cos(\theta_{n-1} - \theta_n) - (s + d) \sin \theta_{i+1}]. \quad (25)$$

Domino $n - 1$ feels, beside the gravitational pull T_{n-1} , a torque from n which slows it down and a torque from $n - 2$ which speeds it up. Generally the equation for domino i has the form

$$\frac{d\omega_i}{dt} = T_i + f_{i-1} a_{i-1} - f_i b_i. \quad (26)$$

The coefficients of the torques follow from the geometry shown in Fig. 1.

$$a_i = h \cos(\theta_i - \theta_{i+1}) - (s + d) \sin \theta_{i+1}, \quad b_i = h \cos(\theta_i - \theta_{i+1}). \quad (27)$$

Note that the first equation (25) is just a special case with $f_n = 0$. Another interesting feature is that $a_i < b_i$. So i gains less from $i - 1$ than $i - 1$ loses to i . Therefore dominoes, falling concertedly, gain less angular momentum than if they would fall independently. This will have a consequence on the application of conservation of angular momentum in the collision process. We come back on this issue in the next section.

We can eliminate the forces from the equation by multiplying (25) with $r_0 = 1$ and the general equation with r_{n-i} and choosing the values of r_j such that

$$r_j = r_{j-1} \frac{a_{n-j}}{b_{n-j}}, \quad (r_0 = 1), \quad \text{or} \quad r_{n-i} = \prod_{j=i}^{n-1} \frac{a_j}{b_j}. \quad (28)$$

Then adding all the equations gives

$$\sum_i r_{n-i} \left[\frac{d\omega_i}{dt} - T_i \right] = \sum_i [f_{i-1} r_{n-i} a_{i-1} - f_i r_{n-i-1} a_i] = 0. \quad (29)$$

Now observe that the recursion for the r_j is identical to that of the w_j as given in (10). With $r_0 = 1$ we may identify $r_j = w_j$. It means that if we multiply (29) with ω_n and replace $r_{n-i}\omega_n$ by ω_i , we recover the conservation of energy in the form

$$\frac{d}{dt} \frac{1}{2} \sum_i \omega_i^2 = \sum_i \omega_i T_i. \quad (30)$$

It is not difficult to write the sum of the torques as the derivative with respect to time of the potential energy, thereby casting the conservation of energy in the standard form. So if conservation of energy holds, the elimination of the forces is superfluous. However, equation (29) is more general and we use it in the treatment of friction.

6 The Collision, second version

We have assumed that in the collision of the head of chain n with the next domino $n+1$ conserves angular momentum. Having a more detailed picture of forces between the sliding dominoes we reconsider this assumption. In this section without friction and in Section 8 with friction. The idea is that in the collision domino n , exerts a impulse on $n+1$ and vice versa with opposite sign. In other words: one has to integrate the equations of motion of the previous section over such a short time that the positions do not change, but that the velocities accumulate a finite difference. However, not only the jump in velocity propagates downwards, also the impulses have to propagate downwards in order to realize these jumps. Denoting the impulses by capital F 's, domino i receives F_i from $i+1$ and F_{i-1} from $i-1$. So we get for the jumps in the rotational velocity

$$\left\{ \begin{array}{l} \omega_{n+1}(0) = F_n a_n, \\ w_1(0) \omega_{n+1}(0) - w_0(\theta_c) \omega_n(\theta_c) = F_{n-1} a_{n-1} - F_n b_n, \\ \dots = \dots \\ w_{n+1-i}(0) \omega_{n+1}(0) - w_{n-i}(\theta_c) \omega_n(\theta_c) = F_{i-1} a_{i-1} - F_i b_i. \end{array} \right. \quad (31)$$

The functions a_i and b_i are the same as those defined in (27). If we would have $a_i = b_i$ we could add all equations and indeed find that the angular total angular momentum is conserved in the collision. But only $a_n = b_n$ since $\theta_{n+1} = 0$. The impulse F_i can be eliminated in the same way as before by multiplying the i th equation with r_{n+1-i} and adding them up. For the coefficient of $\omega_{n+1}(0)$ we get

$$\sum_i^{n+1} r_{n+1-i} w_{n+1-i}(0) = \sum_{j=0}^{n+1} r_j w_j(0) = J_{n+1}, \quad (32)$$

and for the coefficient of $\omega_n(\theta_c)$ one finds with (10)

$$\sum_i^n r_{n+1-i} w_{n-i}(\theta) = \sum_i^n r_{n+1-i} w_{n+1-i}(0) = \sum_{j=1}^n r_j w_j(0) = J_{n+1} - 1. \quad (33)$$

As general relation we get

$$J_{n+1} \omega_{n+1}(0) = (J_{n+1} - 1) \omega_n(\theta_c). \quad (34)$$

In our frictionless case $r_j = w_j$ and therefore $J_{n+1} = I_{n+1}(0)$. So the desired relation reads

$$I_{n+1}(0) \omega_{n+1}(0) = (I_{n+1}(0) - 1) \omega_n(\theta_c) = I_n(\theta_c) \omega_n(\theta_c). \quad (35)$$

We have added the last equality since it smells as a conservation of angular momentum using the effective angular moment of inertia $I(\theta)$. This inertia moment is however linked to the energy and not to the angular momentum. The true angular momentum conservation is given in Section 4. It is also not conservation of kinetic energy. Then the squares of the angular velocities would have to enter. The difference with the earlier relation (23) is that the sum involves the squares of the w 's. This has a notable influence on the asymptotic velocity.

7 Fully Developed Solitons

After a sufficient number of rotations and collisions a stationary state sets in. Then we may identify in the collision law the entry $\omega_{n+1}(0)$ with $\omega_n(0)$. This allows to solve for the stationary $\omega_n(0)$. We use (11) to relate the effective moments of inertia

$$I_n(\theta_c) = \sum_{j=0}^n w_j^2(\theta_c) = \sum_{j=0}^{n-1} w_{j+1}^2(0) + w_n^2(\theta_c) = I_n(0) - w_0^2(0) + w_n^2(\theta_c). \quad (36)$$

For large n the last term vanishes and we may drop the n dependence in I_n . So

$$I(\theta_c) = I(0) - 1. \quad (37)$$

The collision laws thus may be asymptotically written as,

$$I(0) \omega_n(0) = [I(0) - 1] \omega_n(\theta_c). \quad (38)$$

The rotation is governed by the conservation of energy, which we write as

$$I(\theta) \omega_n^2(\theta) + H_n(\theta) = I(0) \omega_n^2(0) + H_n(0). \quad (39)$$

We can use (9) to relate the height function $H_n(\theta_c)$ to its value at $\theta = 0$.

$$H_n(\theta_c) = \sum_j^n [\cos p_j(\theta_c) + \frac{d}{h} \sin p_j(\theta_c)] = H_n(0) - 1 + \cos p_n(\theta_c) + \frac{d}{h} \sin p_n(\theta_c). \quad (40)$$

The limiting value of p_n is the stacking angle θ_∞ . Therefore the difference between the initial and the final potential energy reads

$$H(0) - H(\theta_c) = 1 - \cos \theta_\infty - \frac{d}{h} \sin \theta_\infty \equiv P(h, d, s). \quad (41)$$

We have introduced the function P as the loss in potential energy in the soliton motion. It is the difference between an upright domino and a stacked domino at angle θ_∞ . The functional form reads explicitly

$$P(h, d, s) = \frac{sh - d(s^2 + 2sd)^{1/2}}{h(s + d)}. \quad (42)$$

It is clear that the domino effect does not exist if P is negative, because a domino tilted at the stacking angle has a higher potential energy than an upright domino.

We use (41) in the conservation law for the energy, taken at $\theta = \theta_c$

$$I(\theta_c) \omega_n^2(\theta_c) - I(0) \omega_n^2(0) = P(h, d, s). \quad (43)$$

Solving $\omega_n(0)$ and $\omega_n(\theta_c)$ from (38) and (43) yields

$$\omega_n^2(0) = P(h, d, s) \frac{I(0) - 1}{I(0)}, \quad \omega_n^2(\theta_c) = P(h, d, s) \frac{I(0)}{I(0) - 1}. \quad (44)$$

By and large \sqrt{P} sets the scale for the rotation velocity. The dependence on $I(0)$ is rather weak. For large $I(0)$ it drops out. The minimum value of $I(0)$ is 2 which is reached for large separations.

8 Friction

After all this groundwork it is relatively simple to introduce friction. Let us start with the equation of motion (26). Friction adds a force parallel to the surface of $i + 1$. For the strength of the friction force we assume the law of Amonton-Coulomb [4]

$$f_{\text{friction}} = \mu f, \quad (45)$$

where f is the corresponding perpendicular force. Inclusion of friction means that the coefficients a_i and b_i pick up a frictional component. The associated torques follow from the geometry of Fig. 1. So the values of the a_i and b_i change to

$$\begin{cases} a_i &= h \cos(\theta_i - \theta_{i+1}) - (s + d) \sin \theta_{i+1} - \mu d, \\ b_i &= h \cos(\theta_i - \theta_{i+1}) + \mu h \sin(\theta_i - \theta_{i+1}). \end{cases} \quad (46)$$

Then we may eliminate the forces as before, which again leads to (29). But we cannot identify any longer r_i with w_i . In order to use (29) we must express the accelerations $d\omega_i/dt$ in the head of chain $d\omega_n/dt$. This follows from differentiating (8)

$$\frac{d\omega_i}{dt} = w_{n-i}(\theta_n) \frac{d\omega_n}{dt} + v_{n-i}(\theta) \omega_n^2, \quad (47)$$

with v_i given by

$$v_j(\theta) = \frac{dw_j(\theta)}{d\theta_n}. \quad (48)$$

The v_j can be calculated from the recursion relation, that follows from differentiating (10). Clearly the recursion starts with $v_0 = 0$ (see (47)).

Next we insert (47) into (26) and obtain

$$\left(\sum_j^n r_j w_j \right) \frac{d\omega_n}{dt} = \left(\sum_j^n r_j T_{n-j} \right) - \left(\sum_j^n r_j v_j \right) \omega_n^2. \quad (49)$$

The equation can be transformed into a differential equation for $d\omega_n/d\theta_n$ by dividing (50) by $\omega_n = d\theta_n/dt$

$$\left(\sum_j^n r_j w_j \right) \frac{d\omega_n}{d\theta_n} = \left(\sum_j^n r_j T_{n-j} \right) \frac{1}{\omega_n} - \left(\sum_j^n r_j v_j \right) \omega_n. \quad (50)$$

We use this equation to find ω_n as function of θ_n and then (18) again to calculate the duration of the time between two collisions.

The inclusion of friction in the collision law is even simpler, since relation (34) remains valid, but now with the definitions (46) for a_i and b_i .

9 Thin Dominoes

Sometimes limits help to understand the general behaviour. One of the parameters, which has played so far a modest role, is the aspect ratio d/h . In our formulae it is perfectly possible to take this ratio 0. In practice infinitely thin dominoes are a bit weird, because with paperthin dominoes one has e.g. to worry about friction with the air. In this limit we can vary s/h over the full range from 0 to 1. In Fig. 2 we have plotted the asymptotic velocity as function of the separation s/h . The curve is rather flat with a gradual drop-off towards the large separations. We discuss here the two limits where the separation goes to 0 and where it approaches its maximum $s = h$. Both offer some insight in the overall behavior.

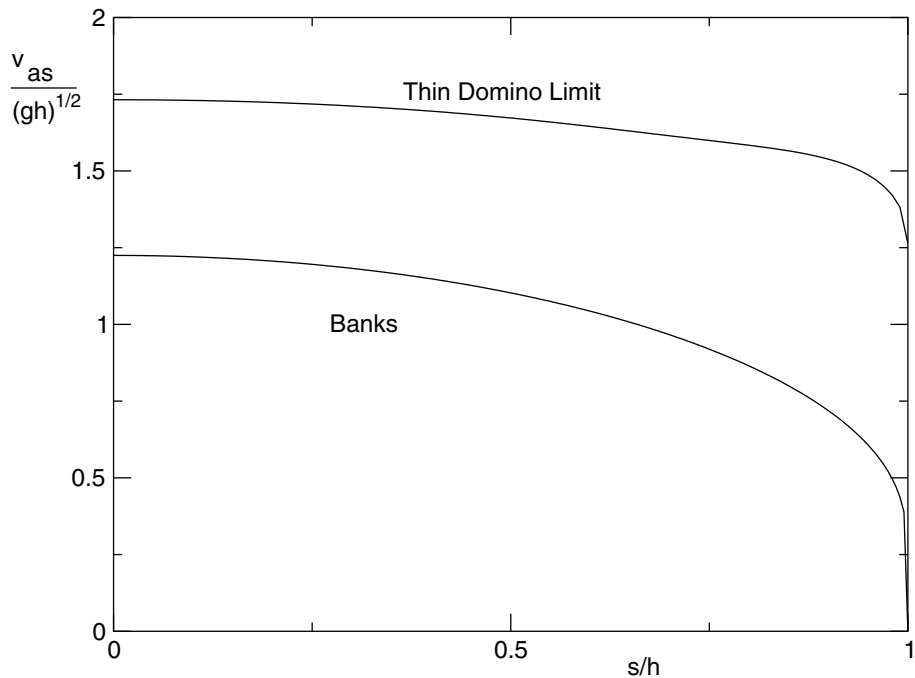


Figure 2: The asymptotic soliton velocity as function of the separation s/h in the thin domino limit. Also is plotted the result of Banks in the same limit.

9.1 Infinitesimal Separation

If the dominoes are narrowly separated, the head of chain rotates only over a small angle $\theta_c = \arcsin(s/h) \simeq s/h$ and the collisions will rapidly succeed each other. The number of dominoes with a tilt angle θ_i between 0 and $\pi/2$ becomes very large and slowly varying with the index i . So a continuum description is appropriate. We first focus on the dependence of $\theta_i(\theta_n)$ on the index i and later comment on the dependence on the weak variation with θ_n (which is confined to the small interval $0 < \theta_n < \theta_c$). We take as coordinate x the distance of domino i .

$$x = i s/h \tag{51}$$

and use $\nu = ns/h$ for the position of the head of the train. Then

$$\theta_i = \theta(x), \quad \theta_{i+1} = \theta(x + dx), \tag{52}$$

with $dx = s/h$. So for $d = 0$ and $s/h \rightarrow 0$ the constraint (1) becomes

$$\sin[\theta(x) - \theta(x + dx)] = dx \cos \theta(x + dx), \tag{53}$$

leading to the differential equation

$$\frac{d\theta(x)}{dx} = -\cos \theta(x), \tag{54}$$

which has the solution

$$\sin \theta(x) = \tanh(\nu - x) \quad \text{or} \quad \theta(x) = \arcsin(\tanh(\nu - x)). \tag{55}$$

Here we have used the boundary condition that $\theta(\nu) = 0$. Not surprisingly we find that the shape of the tilt angles is a function of the difference with respect to the head of the train. The above expression gives the shape of the soliton.

Next we comment on the dependence of this profile on the angle θ_n . As mentioned it can be only weak as the interval for θ_n is narrow. Thus it suffices to know a few derivatives and for that, the interpretation (10) is useful. The behavior of w_j in the continuum limit, follows from the differential form of the recursion relation

$$\frac{dw(x)}{dx} = \sin \theta(x) w(x), \quad (56)$$

with the solution

$$w(x) = \frac{1}{\cosh(\nu - x)} = \cos \theta(x). \quad (57)$$

Note that, not unexpectedly, the form of $w(x)$ follows also from that of $\theta(x)$ by differentiation with respect to ν . Similarly the expression for v_j , as given by (48), can be obtained from differentiation of (57) with respect to ν

$$v(x) = -\frac{\tanh(\nu - x)}{\cosh(\nu - x)} = -\frac{dw(x)}{dx}. \quad (58)$$

What still is needed is the propagation velocity of the soliton, or in the present language: how fast n or ν moves with time. As the foremost domino rotates over a small angle $\theta_c \simeq s/h$, the head of train covers the distance s/h with the rotation velocity ω_n . So the propagation speed equals ω_n . As before, ω_n has to be distilled from the laws of rotation and collision. Since this section is mainly for illustration, we restrict ourselves to the frictionless case.

In the collision law (35) we encounter $\omega_n(\theta_c)$ and $\omega_{n+1}(0)$. Both are linked to $\omega_n(0) = \omega(\nu)$ by

$$\omega_n(\theta_c) = \omega(\nu) + \frac{\partial \omega_n}{\partial \theta_n} s/h, \quad \omega_{n+1}(0) = \omega(\nu) + \frac{\partial \omega}{\partial \nu} s/h. \quad (59)$$

For the derivative with respect to θ_n , we can take advantage of the form (50) which directly gives this derivative. We use that $r_i = w_i$ in the frictionless case. The sums can be performed explicitly in the continuum limit using (57) and (58)

$$\left\{ \begin{array}{l} \frac{s}{h} \sum_j^n w_j^2 = \int_0^\nu dx \frac{1}{\cosh^2(\nu - x)} = \tanh \nu, \\ \frac{s}{h} \sum_j^n w_j T_{n-j} = \int_0^\nu dx \frac{\tanh(\nu - x)}{2 \cosh(\nu - x)} = \frac{\cosh \nu - 1}{2 \cosh \nu}, \\ \frac{s}{h} \sum_j^n v_j w_j = - \int_0^\nu dx w(x) \frac{dw}{dx} = -\frac{\tanh^2 \nu}{2}. \end{array} \right. \quad (60)$$

Therefore the equation for $\partial \omega_n / \partial \theta_n$ becomes

$$\tanh \nu \frac{\partial \omega_n}{\partial \theta_n} = \frac{\cosh \nu - 1}{2 \cosh \nu} \frac{1}{\omega_n} + \frac{1}{2} \tanh^2 \nu \omega_n. \quad (61)$$

With (59) the collision equation has the form

$$\sum_j^n w_j^2(\theta_c) [\omega(\nu) + \frac{\partial \omega_n}{\partial \theta_n} s/h] = [1 + \sum_j^n w_j^2(\theta_c)] [\omega(\nu) + \frac{\partial \omega}{\partial \nu} s/h]. \quad (62)$$

Using (60) we get, to first order in s/h ,

$$\tanh \nu \frac{\partial \omega_n}{\partial \theta_n} = \omega(\nu) + \tanh \nu \frac{\partial \omega}{\partial \nu}. \quad (63)$$

Next we substitute (61) and we obtain the following differential equation for $\omega(\nu)$

$$\tanh \nu \frac{d\omega(\nu)}{d\nu} = \omega(\nu) \left(\frac{1}{2} \tanh^2 \nu - 1 \right) + \frac{\cosh \nu - 1}{2 \cosh \nu} \frac{1}{\omega(\nu)}. \quad (64)$$

This awful looking differential equation has a simple solution

$$\omega^2(\nu) = \frac{\cosh \nu}{\sinh^2 \nu} (\cosh \nu - 1 - \log \cosh \nu). \quad (65)$$

We have chosen the integration constant such that $\omega(\nu)$ vanishes for $\nu = 0$. It starts as

$$\omega(\nu) \simeq \nu/2\sqrt{2} + \dots, \quad \nu \rightarrow 0, \quad (66)$$

and it saturates exponentially fast to the value $\omega(\infty) = 1$ (leading to $v_{as} = \sqrt{3gh}$). Thus we have obtained in the continuum limit a full and explicit solution. It may serve as an illustration for the general discrete case.

9.2 Maximal Separation

On the other side, near maximal separation $s \rightarrow h$, also a simplification occurs. Here the number of dominoes involved in the train is restricted to a few. The head of the train rotates over almost $\pi/2$ before it strikes the next domino. So one comes close to the picture of Banks [2] in which the toppling of the dominoes is a succession of independent events. There is however a difference resulting from the constraint (1). Immediate *after* the collision, the dominoes n and $n + 1$ rotate with equal velocity $\omega_n(\theta_c) = \omega_{n+1}(0)$. This is a consequence of the fact that after the collision, one still has $\theta_{n+1} = 0$. Thus we find for the energy after the collision

$$e_{n+1} = 1 + 2\omega_{n+1}^2(0). \quad (67)$$

All other dominoes have fallen down and domino $n + 1$ is still upright (the 1 in (67)). Once $n+1$ starts rotating, the value of ω_n rapidly drops down to 0. Inspecting recursion (10), with $p_0(\theta_{n+1}) = \theta_{n+1}$ and $p_1(\theta_{n+1}) = \theta_n$, one sees that the factor $\cos(\theta_n - \theta_{n+1}) \simeq \cos(\theta_c)$ is very close to 0. The ratio approaches

$$\frac{s \sin(\theta_{n+1})}{h \cos(\theta_n - \theta_{n+1})} \rightarrow \frac{s \sin(\theta_{n+1})}{h \cos(\pi/2 - \theta_{n+1})} = \frac{s}{h}. \quad (68)$$

So $w_1 \rightarrow 0$ and indeed domino n comes to a halt; it has to, since it has reached the floor. This has an effect on the moment of inertia $I(\theta_{n+1})$ defined in (15). Immediately after the collision the value of $I(\theta_{n+1})$ equals 2, being the sum of $w_{n+1}^2 = 1$ and $w_n^2 = 1$. A small angle further it has dropped to 1, since w_n drops to 0. As the energy is conserved the kinetic energy of domino n is transferred to domino $n + 1$. So ω_{n+1} rises by a factor $\sqrt{2}$ in a short interval. Therefore we start the integration of the time after this sudden increase, using the conservation law for the energy

$$\omega_{n+1}^2(\theta_{n+1}) + \cos \theta_{n+1} = 1 + 2\omega_{n+1}^2(0). \quad (69)$$

In particular we have the relation for $\theta_c \simeq \pi/2$

$$\omega_{n+1}^2(\theta_c) = 1 + 2\omega_{n+1}^2(0). \quad (70)$$

The collision law for this degenerate case becomes

$$\omega_{n+1}(0) = \omega_n(\theta_c)/2. \quad (71)$$

Note that since the w_j are either 1 or 0, there is no difference between the proposal by Shaw (see (23)) and ours (see (35)).

The stationary state is obtained by the identification $\omega_n(\theta_c) = \omega_{n+1}(\theta_c)$. Combining (70) and (71) then yields $\omega_{n+1}(0) = 1/\sqrt{2}$. Thus the time integral for the interval becomes in the stationary state

$$t = \int_0^{\pi/2} d\theta \frac{1}{\sqrt{2 - \cos \theta}} = 1.37 \quad (72)$$

The reciprocal yields the asymptotic soliton speed $v_{\text{as}} = 0.73 * \sqrt{3} = 1.26$.

The story of thin dominoes gives a warning on the numerical integration scheme. For small separations we need many intervals before the asymptotic behaviour has set in. On the other hand we do not need many points in the integration for the time of a rotation. For wide separations it is the opposite: only a few intervals are needed for the asymptotic behavior, but we have to perform the time integration with care. The factor $I(\theta)$ in the energy law is rapidly varying for small θ . So we need many points for small θ to be accurate.

In Fig. 2 we have also plotted the curve due to Banks [2] in the limit of thin dominoes. The difference is due to the collision law, for which Banks takes

$$\omega_{n+1}(0) = \cos \theta_c \omega_n(\theta_c). \quad (73)$$

The factor $\cos \theta_c$ accounts for the horizontal component of the linear momentum. For large separations this gives quite a different value, since the transmission of linear momentum becomes inefficient. For small separations the transmission is nearly perfect (as becomes our collision law). The conservation of energy of a single rotating domino reads

$$\omega^2(\theta) + \cos \theta = \omega^2(0) + 1. \quad (74)$$

For the stationary state we insert (73) into (74) and get

$$\omega^2(\theta_c) = \frac{1 - \cos \theta_c}{1 - \cos^2 \theta_c}. \quad (75)$$

For small θ_c , the value $\omega(\theta_c)$ approaches $1/\sqrt{2}$, which is again substantially smaller than our limiting value 1. The reason is that the dominoes, which keep leaning onto each other and onto the head of the train, speed up the soliton.

10 Calculations and Limitations

For the frictionless case we can use the formulae of Section 7, i.e. first calculate the asymptotic value of $\omega(0)$ and then integrate the rotation equation to find the time between two collisions and thus the asymptotic soliton speed. With friction we must

s/h	Shaw	frictionless	$\mu = 0.1$	$\mu = 0.2$	$\mu = 0.3$
0.1	3.64568	2.23469	1.82095	1.51695	1.28221
0.2	3.06742	1.95534	1.66019	1.43423	1.25452
0.3	2.74686	1.82515	1.56987	1.37279	1.21498
0.4	2.50849	1.74231	1.50459	1.32193	1.17605
0.5	2.30183	1.67865	1.44771	1.27272	1.13420
0.6	2.10337	1.62447	1.39204	1.22009	1.08609
0.7	1.89899	1.57824	1.33347	1.15958	1.02745
0.8	1.68680	1.53779	1.26196	1.07970	0.94711
0.9	1.47984	1.47984	1.15267	0.95822	0.82681

Table 1: The asymptotic soliton speed ($d/h = 0.179$) for the collision law of Shaw and for various degrees of friction with the collision law (34).

iteratively find $\omega(0)$, by trying a value of $\omega(0)$, then solve equation (50) for $\omega(\theta_c)$ and finally apply the collision law in order to see whether we come back to our trial $\omega(0)$. A form of iteration is to start the train with one domino and an arbitrary initial $\omega_0(0)$ and let the train grow longer such that an asymptotic pattern develops.

In Table 1 we have summarized the results. The thickness to height ratio is set at $d/h = 0.179$ since this is the only value on which experiments [6] are reported. The first column gives the separation s/h , the second the soliton speed using Shaw's collision law and the third gives the results for ours (35). In the subsequent columns the influence of the friction is indicated. Note that the reduction of the speed due to the change of the collision law is larger than that of modest friction. The curves corresponding to these values are shown in Fig. 3, which also contains the experiments of Maclachlan et al. They suggest that the soliton speed diverges for short distances, while we find a maximum. Their values seem to correspond best with the friction coefficient $\mu = 0.3$. We found empirically the value $\mu = 0.2$, by estimating the angle of the supporting table at which dominoes start to slide over each other.

In order to make the behavior of fully developed solitons more transparent, we may introduce, for frictionless dominoes, the average

$$\frac{1}{\langle \omega \rangle} = \frac{1}{\theta_c} \int_0^{\theta_c} \frac{d\theta}{\omega(\theta)}, \quad (76)$$

with $\omega(\theta)$ the solution of (39). This average is a number close to $1/\sqrt{P}$ (with P defined in (42)), since the integrand varies from a value slightly larger than $1/\sqrt{P}$ to a value slightly less than $1/\sqrt{P}$. Then we get for the asymptotic soliton speed the formula

$$\frac{v_{as}}{\sqrt{gh}} = Q(h, d, s) \frac{\langle \omega \rangle}{\sqrt{P(h, d, s)}}, \quad (77)$$

where the factor Q is given by

$$Q(h, d, s) = \left(\frac{3}{1 + d^2/h^2} \right)^{1/2} \frac{(s + d) \sqrt{P(h, d, s)}}{h \arcsin(s/h)}. \quad (78)$$

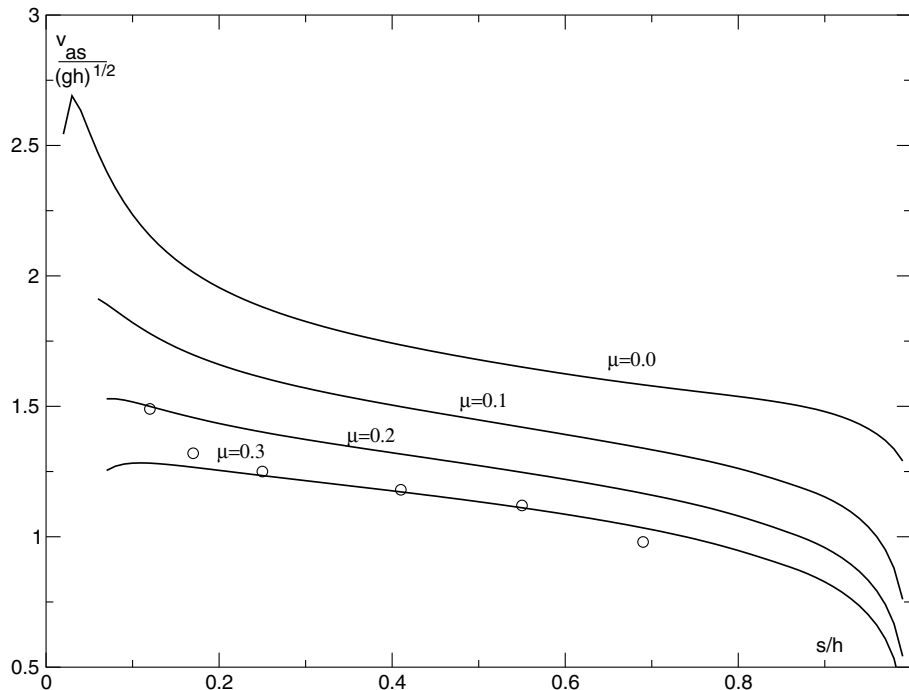


Figure 3: The influence of friction on the asymptotic soliton speed for the aspect ratio $d/h = 0.179$. The dots are the experimental values of Maclachlan et al. [6].

Here we have reinstated the factor I/mgh in order to include in this formula, all the factors that contribute to the velocity. The factor Q is shown as function of s/h for various d/h in Fig. 4. One may consider Q as the main factor determining the dependence of the soliton speed on the parameters of the problem. The fraction in (77) is a refinement which requires a detailed calculation. We found that this fraction is virtually independent of the aspect ratio d/h . It stays close to 1 for the the major part of the range of practical separations. Only around the already “unworkable” separation $s/h = 0.9$ the value has increased some 10%. A good indicator for the behavior is the curve for the frictionless thin dominoes which is the product of the fraction and $Q = \sqrt{3} s / (h \arcsin(s/h))$.

We mentioned in Section 7 that the function P as given by (42) has to be positive for the existence of the domino effect. This gives a bound on the minimal distance s/h , which can be cast in the form

$$\frac{s}{h} > \frac{2(d/h)^3}{1 - (d/h)^2}. \quad (79)$$

Separations smaller than the value of (79) do not show the domino effect and slightly above that limit the train has difficulty to develop. The reason is that after a while, too many dominoes of the train get tilt angles, which have a higher potential energy than an upright domino. Ultimately the fraction of these dominoes in the train loses out against the dominoes at the end of the train, which are tilted at the stacking angle (with a potential energy *lower* than an upright domino). One can overcome this barrier by starting with an unreasonable high initial $\omega_0(0)$. So (79) is the true theoretical limit, but in practice the domino effect will not start for slightly larger values of s/h .

Another limitation of the *theory* is at the other side. The dominoes at the end of the train are tilted at the stacking angle θ_∞ *provided* the height h is sufficiently large.

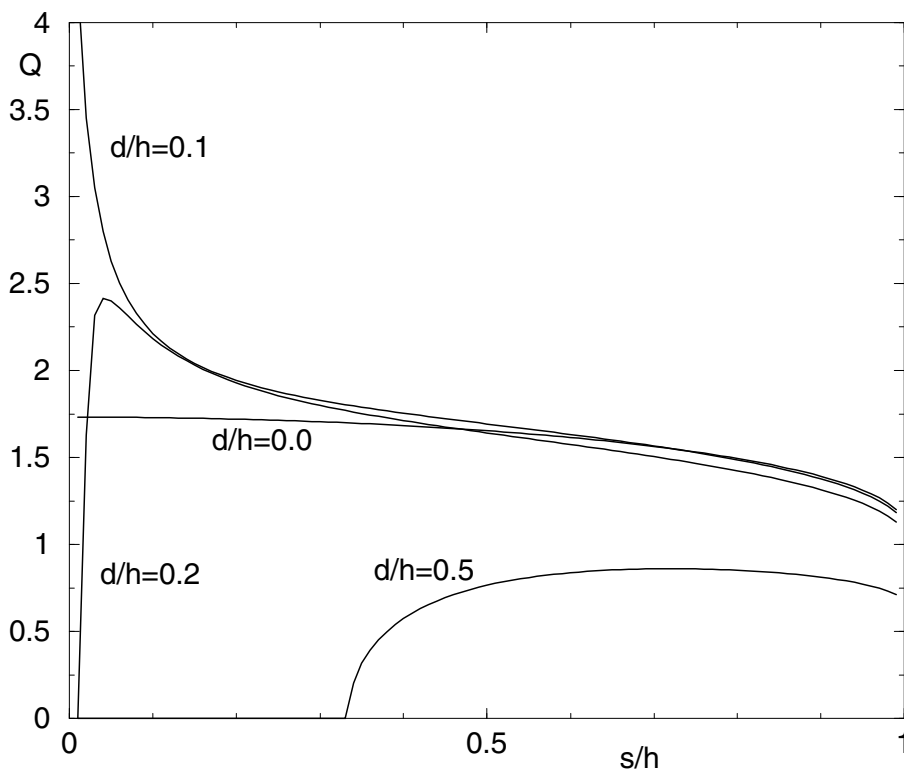


Figure 4: The function $Q(s, d, h)$ as defined in (78) for various d/h .

The condition is

$$h^2 > (s + d)^2 - d^2. \quad (80)$$

For smaller h the dominoes fall flat on the supporting table. (80) is satisfied for

$$s/h < \sqrt{1 + (d/h)^2} - d/h. \quad (81)$$

Beyond this value the train is actually shorter than blind application of the formulae would suggest. It is not so interesting to sort out what precisely happens if (81) is violated, since then the no-slip condition for the dominoes is highly questionable. For such wide separations the force on the struck domino has hardly a torque to rotate it. It rather induces the rotation axis to slide along the table. In fact, as a practical limitation, we look to the height of impact. If it is above the center of mass of the struck domino, it will start to rotate and below that value, it may slip if the friction with the supporting table is not large enough. This criterion yields the limit to the distance

$$s/h < \sqrt{3}/2 = 0.87, \quad (82)$$

which is already a large separation, not far from the limit set by (81) for $d/h = 0.179$.

For very thin dominoes (79) is hardly a limitation. However, (79) and (82) form a window of separations for the existence of the domino effect, which depends on the thickness d/h . This window narrows down to zero and the domino effect disappears for

$$h^3 < hd^2 + 4d^3/\sqrt{3}, \quad \text{or} \quad d/h < 0.66 \quad (83)$$

This estimate is larger than the one given by Freericks [5]. Friction also makes the excluded interval larger. For $\mu = 0.2$ we have not found a domino effect for $s/h < 0.07$,

which is, for $d/h = 0.179$, about 7 times the theoretical limit. So our estimate (83) for the upper thickness is still too optimistic.

11 Discussion

We have studied the toppling of a row of equally spaced dominoes under the assumptions that the dominoes only rotate and that they keep leaning onto each other after a collision with the next one. By and large we follow the treatment of Shaw [3], who introduced the constraint (1), which synchronizes the motion of train of toppling dominoes. By analyzing the mutual forces between the dominoes, we have corrected his collision law and we could also account for the effect of friction between the dominoes. The correction of the collision law is more important than the influence of friction, given the small friction coefficient between dominoes. The limit of thin dominoes $d/h \rightarrow 0$ leads to a completely tractable model. For large separations we encounter a situation which resembles the viewpoint of Banks [2], seeing the toppling as a succession of independent events. However his collision law differs substantially from ours and cannot be reconciled with the force picture that we develop. We give a formula (77), which displays explicitly the main dependence of the soliton speed on the parameters of the problem. The maximum speed which can be reached, appears close to the closest separation for which the domino effect exists.

The assumptions, on which our calculation are based, are the no-slip condition and the constraint (1). One can help the no-slip condition by increasing the friction with the supporting table (putting them on sandpaper as Walker [1] does). If the no-slip condition is violated, it is the end of the domino effect as the dominoes are kicked over with the wrong rotation. We argue that this will happen when the falling domino hits the next one below its center of mass.

The constraint (1) is implied by the assumption that the collision is fully inelastic. This assumption is supported by slow motion pictures of the effect, which show that the dominoes indeed lean onto each other while falling. It is an interesting question what happens, if the collision would be less inelastic. The extreme opposite, fully elastic collisions, yields an ever increasing soliton speed. A falling domino increases its rotation velocity as soon as its center of mass goes down. If this is fully transmitted to the next domino, the rotation velocity keeps increasing. In that case friction can not play a role since the dominoes do not touch each other. In the less extreme case of partially inelastic collisions, the dominoes also rotate without contact, but friction can play a role during the collision. As Fig. 1 indicates, friction always rotates the mutual impulse such, that the torque on the next one decreases and that the reaction torque increases. This will slow down the train and a stationary state can develop. Therefore it would be interesting to experiment with dominoes of different making (e.g. steel) to see the increase in the soliton speed.

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