Solutionset 6.

Excercise 1

We want to show that the Hamiltonian of the free electromagnetic field

$$\mathcal{H}_\gamma = \sum_{\mathbf{k},\boldsymbol{\lambda}} \hbar\omega_k (a^\dagger_{\mathbf{k}\boldsymbol{\lambda}} a_{\mathbf{k}\boldsymbol{\lambda}} + \frac{1}{2}) ,$$

can be obtained by evaluating the expression

$$\mathcal{H}_\gamma = \frac{1}{8\pi} \int d^3\mathbf{x} \, (\mathbf{E}^2 + \mathbf{B}^2) ,$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic field Heisenberg operators.

The quantized vector potential of the free field in the Heisenberg representation is given by

$$\mathbf{A}(\mathbf{x},t) = \sum_{\mathbf{k}\boldsymbol{\lambda}} \left(\frac{2\pi\hbar c^2}{\omega_k V}\right)^{1/2} \left(\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} a_{\mathbf{k}\boldsymbol{\lambda}}(t)\boldsymbol{\lambda} + \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}} a^\dagger_{\mathbf{k}\boldsymbol{\lambda}}(t)\boldsymbol{\lambda}^*\right) ,$$

where $a_{\mathbf{k}\boldsymbol{\lambda}}(t) = \mathrm{e}^{-i\omega_k t} a_{\mathbf{k}\boldsymbol{\lambda}}$, $a^\dagger_{\mathbf{k}\boldsymbol{\lambda}}(t) = \mathrm{e}^{i\omega_k t} a^\dagger_{\mathbf{k}\boldsymbol{\lambda}}$, and $\{a_{\mathbf{k}\boldsymbol{\lambda}}\}$, $\{a^\dagger_{\mathbf{k}\boldsymbol{\lambda}}\}$ satisfy the commutation relations we discussed in class.

Let us rewrite $\mathbf{A}$ in a more compact form:

$$\mathbf{A}(\mathbf{x},t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}\boldsymbol{\lambda}} \left(\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} A_{\mathbf{k}\boldsymbol{\lambda}}(t)\boldsymbol{\lambda} + \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}} A^\dagger_{\mathbf{k}\boldsymbol{\lambda}}(t)\boldsymbol{\lambda}^*\right) ,$$

where $A_{\mathbf{k}\boldsymbol{\lambda}}(t) = \left(\frac{2\pi\hbar c^2}{\omega_k}\right)^{1/2} a_{\mathbf{k}\boldsymbol{\lambda}}(t)$.

The electric and magnetic field Heisenberg operators are given by

$$\mathbf{E}(\mathbf{x},t) = \frac{1}{\sqrt{V}} \frac{i}{c} \sum_{\mathbf{k}\boldsymbol{\lambda}} \omega_k \left(\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} A_{\mathbf{k}\boldsymbol{\lambda}}(t)\boldsymbol{\lambda} - \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}} A^\dagger_{\mathbf{k}\boldsymbol{\lambda}}(t)\boldsymbol{\lambda}^*\right)$$
$$\mathbf{B}(\mathbf{x},t) = \frac{1}{\sqrt{V}} i \sum_{\mathbf{k}\boldsymbol{\lambda}} \left(\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} A_{\mathbf{k}\boldsymbol{\lambda}}(t)\mathbf{k}\times\boldsymbol{\lambda} - \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}} A^\dagger_{\mathbf{k}\boldsymbol{\lambda}}(t)\mathbf{k}\times\boldsymbol{\lambda}^*\right) ,$$

which can be obtained from $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$.

Let us compute $\mathbf{E}^2$:

$$\mathbf{E}^2 = -\frac{1}{V}\frac{1}{c^2}\sum_{\mathbf{k}\boldsymbol{\lambda}}\sum_{\mathbf{k}'\boldsymbol{\lambda}'}\omega_k\omega_{k'}\left(\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}A_{\mathbf{k}\boldsymbol{\lambda}}(t)\boldsymbol{\lambda} - \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}}A^{\dagger}_{\mathbf{k}\boldsymbol{\lambda}}(t)\boldsymbol{\lambda}^*\right)\cdot\left(\mathrm{e}^{i\mathbf{k}'\cdot\mathbf{x}}A_{\mathbf{k}'\boldsymbol{\lambda}'}(t)\boldsymbol{\lambda}' - \mathrm{e}^{-i\mathbf{k}'\cdot\mathbf{x}}A^{\dagger}_{\mathbf{k}'\boldsymbol{\lambda}'}(t)\boldsymbol{\lambda}'^*\right) .$$

By taking $\int \mathrm{d}\mathbf{x}\ \mathbf{E}^2$, and using $\frac{1}{V}\int \mathrm{d}\mathbf{x}\ \mathrm{e}^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}} = \delta_{\mathbf{k},-\mathbf{k}'}$ and $\frac{1}{V}\int \mathrm{d}\mathbf{x}\ \mathrm{e}^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} = \delta_{\mathbf{k},\mathbf{k}'}$:

$$\begin{aligned}\int \mathrm{d}\mathbf{x}\ \mathbf{E}^2 = &-\frac{1}{c^2}\sum_{\mathbf{k}\boldsymbol{\lambda}\boldsymbol{\lambda}'}\omega_k^2\left(A_{\mathbf{k}\boldsymbol{\lambda}}A_{-\mathbf{k}\boldsymbol{\lambda}'}\mathrm{e}^{-i2\omega_k t}\boldsymbol{\lambda}\cdot\boldsymbol{\lambda}' + A^{\dagger}_{\mathbf{k}\boldsymbol{\lambda}}A^{\dagger}_{-\mathbf{k}\boldsymbol{\lambda}'}\mathrm{e}^{i2\omega_k t}\boldsymbol{\lambda}^*\cdot\boldsymbol{\lambda}'^*\right)\\ &-\frac{1}{c^2}\sum_{\mathbf{k}\boldsymbol{\lambda}\boldsymbol{\lambda}'}\omega_k^2\left(-A_{\mathbf{k}\boldsymbol{\lambda}}A^{\dagger}_{\mathbf{k}\boldsymbol{\lambda}'}\,\boldsymbol{\lambda}\cdot\boldsymbol{\lambda}'^* - A^{\dagger}_{\mathbf{k}\boldsymbol{\lambda}}A_{\mathbf{k}\boldsymbol{\lambda}'}\,\boldsymbol{\lambda}^*\cdot\boldsymbol{\lambda}'\right) .\end{aligned}$$

In the first sum, we have the terms with $\mathbf{k}' = -\mathbf{k}$ and $\boldsymbol{\lambda}$ is perpendicular to $\mathbf{k}$, while $\boldsymbol{\lambda}'$ is perpendicular to $-\mathbf{k}$, so the two sets $\{\boldsymbol{\lambda}\}$ and $\{\boldsymbol{\lambda}'\}$ may not coincide.

In the second sum, we have the terms with $\mathbf{k}' = \mathbf{k}$ so that $\{\boldsymbol{\lambda}\}$ and $\{\boldsymbol{\lambda}'\}$ are the same set of vectors and $\boldsymbol{\lambda}\cdot\boldsymbol{\lambda}'^* = |\boldsymbol{\lambda}|^2\delta_{\boldsymbol{\lambda}\boldsymbol{\lambda}'} = \delta_{\boldsymbol{\lambda}\boldsymbol{\lambda}'}$. Therefore, we have

$$\begin{aligned}\int \mathrm{d}\mathbf{x}\ \mathbf{E}^2 = &-\frac{1}{c^2}\sum_{\mathbf{k}\boldsymbol{\lambda}\boldsymbol{\lambda}'}\omega_k^2\left(A_{\mathbf{k}\boldsymbol{\lambda}}A_{-\mathbf{k}\boldsymbol{\lambda}'}\,\mathrm{e}^{-i2\omega_k t}\boldsymbol{\lambda}\cdot\boldsymbol{\lambda}' + A^{\dagger}_{\mathbf{k}\boldsymbol{\lambda}}A^{\dagger}_{-\mathbf{k}\boldsymbol{\lambda}'}\,\mathrm{e}^{i2\omega_k t}\boldsymbol{\lambda}^*\cdot\boldsymbol{\lambda}'^*\right)\\ &+\frac{1}{c^2}\sum_{\mathbf{k}\boldsymbol{\lambda}}\omega_k^2\left(A_{\mathbf{k}\boldsymbol{\lambda}}A^{\dagger}_{\mathbf{k}\boldsymbol{\lambda}} + A^{\dagger}_{\mathbf{k}\boldsymbol{\lambda}}A_{\mathbf{k}\boldsymbol{\lambda}}\right) .\end{aligned}$$

Similarly, we compute $\mathbf{B}^2$:

$$\begin{aligned}\mathbf{B}^2 = -\frac{1}{V}\sum_{\mathbf{k}\boldsymbol{\lambda}}\sum_{\mathbf{k}'\boldsymbol{\lambda}'}\ &\left(\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}A_{\mathbf{k}\boldsymbol{\lambda}}(t)\,\mathbf{k}\times\boldsymbol{\lambda} - \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}}A^{\dagger}_{\mathbf{k}\boldsymbol{\lambda}}(t)\,\mathbf{k}\times\boldsymbol{\lambda}^*\right)\cdot\\ &\cdot\left(\mathrm{e}^{i\mathbf{k}'\cdot\mathbf{x}}A_{\mathbf{k}'\boldsymbol{\lambda}'}(t)\,\mathbf{k}'\times\boldsymbol{\lambda}' - \mathrm{e}^{-i\mathbf{k}'\cdot\mathbf{x}}A^{\dagger}_{\mathbf{k}'\boldsymbol{\lambda}'}(t)\,\mathbf{k}'\times\boldsymbol{\lambda}'^*\right) .\end{aligned}$$

By taking $\int \mathrm{d}\mathbf{x}\ \mathbf{B}^2$, we have

$$\begin{aligned}\int \mathrm{d}\mathbf{x}\ \mathbf{B}^2 = &-\sum_{\mathbf{k}\boldsymbol{\lambda}\boldsymbol{\lambda}'}\left(-A_{\mathbf{k}\boldsymbol{\lambda}}A_{-\mathbf{k}\boldsymbol{\lambda}'}\,\mathrm{e}^{-i2\omega_k t}(\mathbf{k}\times\boldsymbol{\lambda})\cdot(\mathbf{k}\times\boldsymbol{\lambda}') - A^{\dagger}_{\mathbf{k}\boldsymbol{\lambda}}A^{\dagger}_{-\mathbf{k}\boldsymbol{\lambda}'}\,\mathrm{e}^{i2\omega_k t}(\mathbf{k}\times\boldsymbol{\lambda}^*)\cdot(\mathbf{k}\times\boldsymbol{\lambda}'^*)\right)\\ &-\sum_{\mathbf{k}\boldsymbol{\lambda}\boldsymbol{\lambda}'}\left(-A_{\mathbf{k}\boldsymbol{\lambda}}A^{\dagger}_{\mathbf{k}\boldsymbol{\lambda}'}(\mathbf{k}\times\boldsymbol{\lambda})\cdot(\mathbf{k}\times\boldsymbol{\lambda}'^*) - A^{\dagger}_{\mathbf{k}\boldsymbol{\lambda}}A_{\mathbf{k}\boldsymbol{\lambda}'}(\mathbf{k}\times\boldsymbol{\lambda}^*)\cdot(\mathbf{k}\times\boldsymbol{\lambda}')\right) .\end{aligned}$$

Again, in the first sum, the term $\mathbf{k}' = -\mathbf{k}$ survived and the set $\{\boldsymbol{\lambda}'\}$ is a different set than $\{\boldsymbol{\lambda}\}$. In the second sum, the terms $\mathbf{k}' = \mathbf{k}$ survived and $\{\boldsymbol{\lambda}'\}$ and $\{\boldsymbol{\lambda}\}$ are the same set of vectors.

Note that $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$ so that, for instance,

$$(\mathbf{k}\times\boldsymbol{\lambda}^*)\cdot(\mathbf{k}\times\boldsymbol{\lambda}') = k^2\boldsymbol{\lambda}^*\cdot\boldsymbol{\lambda}' - (\mathbf{k}\cdot\boldsymbol{\lambda}')(\mathbf{k}\cdot\boldsymbol{\lambda}^*) = k^2\boldsymbol{\lambda}^*\cdot\boldsymbol{\lambda}'$$

since $\{\boldsymbol{\lambda}'\}$ and $\{\boldsymbol{\lambda}\}$ are perpendicular to $\mathbf{k}$. In the second sum, $k^2\boldsymbol{\lambda}^*\cdot\boldsymbol{\lambda}' = k^2\delta_{\boldsymbol{\lambda}\boldsymbol{\lambda}'} = \frac{\omega_k^2}{c^2}\,\delta_{\boldsymbol{\lambda}\boldsymbol{\lambda}'}$.

Therefore, we have

$$\int \mathrm{d}\mathbf{x}\, \mathbf{B}^2 = \frac{1}{c^2} \sum_{\mathbf{k}\boldsymbol{\lambda}\boldsymbol{\lambda}'} \omega_k^2 \left(A_{\mathbf{k}\boldsymbol{\lambda}} A_{-\mathbf{k}\boldsymbol{\lambda}'}\, \mathrm{e}^{-i2\omega_k t} (\boldsymbol{\lambda}\cdot\boldsymbol{\lambda}') + A^\dagger_{\mathbf{k}\boldsymbol{\lambda}} A^\dagger_{-\mathbf{k}\boldsymbol{\lambda}'}\, \mathrm{e}^{i2\omega_k t} (\boldsymbol{\lambda}^*\cdot\boldsymbol{\lambda}'^*) \right)$$
$$+ \frac{1}{c^2} \sum_{\mathbf{k}\boldsymbol{\lambda}} \omega_k^2 \left(A_{\mathbf{k}\boldsymbol{\lambda}} A^\dagger_{\mathbf{k}\boldsymbol{\lambda}} + A^\dagger_{\mathbf{k}\boldsymbol{\lambda}} A_{\mathbf{k}\boldsymbol{\lambda}} \right) .$$

The sum $\sum_{\mathbf{k}\boldsymbol{\lambda}\boldsymbol{\lambda}'}$ appears in both $\int \mathbf{E}^2$ and $\int \mathbf{B}^2$ but with opposite sign, so this sum cancels when computing $\frac{1}{8\pi}\int(\mathbf{E}^2+\mathbf{B}^2)$. Finally, we obtain

$$\begin{aligned}
\frac{1}{8\pi}\int(\mathbf{E}^2+\mathbf{B}^2) &= \frac{1}{8\pi}\frac{2}{c^2} \sum_{\mathbf{k}\boldsymbol{\lambda}} \omega_k^2 \left(A_{\mathbf{k}\boldsymbol{\lambda}} A^\dagger_{\mathbf{k}\boldsymbol{\lambda}} + A^\dagger_{\mathbf{k}\boldsymbol{\lambda}} A_{\mathbf{k}\boldsymbol{\lambda}} \right) \\
&= \frac{1}{4\pi c^2} \sum_{\mathbf{k}\boldsymbol{\lambda}} \omega_k^2 \left(\frac{2\pi\hbar c^2}{\omega_k} \right) \left(a_{\mathbf{k}\boldsymbol{\lambda}} a^\dagger_{\mathbf{k}\boldsymbol{\lambda}} + a^\dagger_{\mathbf{k}\boldsymbol{\lambda}} a_{\mathbf{k}\boldsymbol{\lambda}} \right) \\
&= \frac{1}{2} \sum_{\mathbf{k}\boldsymbol{\lambda}} \hbar\omega_k \left(a_{\mathbf{k}\boldsymbol{\lambda}} a^\dagger_{\mathbf{k}\boldsymbol{\lambda}} + a^\dagger_{\mathbf{k}\boldsymbol{\lambda}} a_{\mathbf{k}\boldsymbol{\lambda}} \right) \\
&= \frac{1}{2} \sum_{\mathbf{k}\boldsymbol{\lambda}} \hbar\omega_k \left(2a^\dagger_{\mathbf{k}\boldsymbol{\lambda}} a_{\mathbf{k}\boldsymbol{\lambda}} + a_{\mathbf{k}\boldsymbol{\lambda}} a^\dagger_{\mathbf{k}\boldsymbol{\lambda}} - a^\dagger_{\mathbf{k}\boldsymbol{\lambda}} a_{\mathbf{k}\boldsymbol{\lambda}} \right) \\
&= \frac{1}{2} \sum_{\mathbf{k}\boldsymbol{\lambda}} \hbar\omega_k \left(2a^\dagger_{\mathbf{k}\boldsymbol{\lambda}} a_{\mathbf{k}\boldsymbol{\lambda}} + 1 \right) \\
&= \sum_{\mathbf{k}\boldsymbol{\lambda}} \hbar\omega_k \left(a^\dagger_{\mathbf{k}\boldsymbol{\lambda}} a_{\mathbf{k}\boldsymbol{\lambda}} + \frac{1}{2} \right) ,
\end{aligned}$$

since $[a_{\mathbf{k}\boldsymbol{\lambda}}, a^\dagger_{\mathbf{k}\boldsymbol{\lambda}}] = 1$.

Excercise 2

(a) To diagonalize the Hamiltonian

$$\hat{\mathcal{H}}_{ph} = \sum_n \left\{ \frac{\hat{p}_n^2}{2m} + \frac{\gamma}{2}(\hat{u}_{n+1} - \hat{u}_n)^2 \right\}$$

with $\hat{u}_n$ and $\hat{p}_n$ fulfilling the canonical relation $[\hat{u}_n, \hat{p}_m] = i\delta_{nm}$ ($\hbar = 1$), we first go to momentum space

$$\hat{p}_n = \int_k \hat{p}_k e^{ikr_n},$$
$$\hat{u}_n = \int_k \hat{u}_k e^{-ikr_n},$$

with $[\hat{u}_k, \hat{p}_{k'}] = i\delta(k - k')$. For abbreviation we have used $\int_k := \frac{a}{2\pi}\int_{-\pi/a}^{\pi/a} dk$. In momentum space the Hamiltonian becomes

$$\hat{\mathcal{H}}_{ph} = \int_k \left\{ \frac{1}{2m}\hat{p}_k\hat{p}_{-k} + \gamma(1 - \cos(ka))\hat{u}_k\hat{u}_{-k} \right\},$$

where we have made use of the fact that $\hat{\mathcal{H}}_{ph} = \int_k h_k = \int_k (h_k + h_{-k})/2$. Now we can express the displacement and conjugated momentum operators via canonical bosonic operators a_k and $a_k^\dagger$ ($[a_k, a_{k'}^\dagger] = \delta_{k,k'}$, $[a_k, a_{k'}] = [a_k^\dagger, a_{k'}^\dagger] = 0$) due to

$$\hat{u}_k = \frac{1}{\sqrt{2m\omega_k}}(a_k + a_{-k}^\dagger),$$
$$\hat{p}_k = i\sqrt{\frac{2m\omega_k}{2}}(a_k^\dagger - a_{-k}),$$

with $\omega_k^2 = 2\frac{\gamma}{m}(1 - \cos(ka))$ to obtain

$$\hat{\mathcal{H}}_{ph} = \int_k \omega_k (a_k^\dagger a_k + \frac{1}{2}).$$

For small k we find the linear dispersion $\omega_k \approx c_s|k|$, with $c_s = a\sqrt{\frac{\gamma}{m}}$ the sound velocity of the acoustic phonon. Finally, let us calculate the ground state energy:

$$E_0 = \int_k \omega_k/2 = \frac{a}{2\pi}\int_{-\pi/a}^{\pi/a} \omega_k/2 = \frac{a}{2\pi}\sqrt{\frac{2\gamma}{m}}\int_0^{\pi/a} dk\sqrt{1 - \cos(ka)} = \frac{2}{\pi}\sqrt{\frac{\gamma}{m}}.$$

(b) First we use the translational symmetry to show that

$$\mathcal{H}' = \gamma' \sum_n (u_{n+1} - u_n)^3 = 3\gamma' \sum_n (u_{n+1}u_n^2 - u_{n+1}^2 u_n).$$

Now we transform to momentum space, $u_n = \int_k u_k \exp(-ikr_n)$, and directly obtain

$$\begin{aligned}\mathcal{H}' &= 3\gamma' \int_k \int_{k'} \int_{k''} u_k u_{k'} u_{k''} e^{-ika}(1 - e^{-ik'a})\delta(k + k' + k'') \\ &= \frac{3\gamma'}{\sqrt{2m}^3} \int_k \int_{k'} \int_{k''} \frac{1}{\sqrt{\omega_k \omega_{k'} \omega_{k''}}}(a_k + a^\dagger_{-k})(a_{k'} + a^\dagger_{-k'})(a_{k''} + a^\dagger_{-k''})\delta(k + k' + k'').\end{aligned}$$

Since the anharmonic terms do not violate the translational symmetry, the total momentum is still conserved. If you want, you can also show that the commutator $[P, \mathcal{H}']$ gives zero.

Excercise 3

Calculate the lifetime of the hydrogen atom in its $2p$ state.

The initial state for the atom and the field is $|I\rangle = |nlm\rangle \otimes |0\rangle$ with $|nlm\rangle = |21m\rangle$, the $2p$ state of the atom, and $|0\rangle$ the vacuum of the electromagnetic field.

The final state is given by $|F\rangle = |100\rangle \otimes |\mathbf{k}\boldsymbol{\lambda}\rangle$, where $|100\rangle$ is the ground state of the atom and $|\mathbf{k}\boldsymbol{\lambda}\rangle$ is the state of one $\mathbf{k}\boldsymbol{\lambda}$ photon.

The initial and final energies are $E_I = E_{nlm} = E_{21m}$ and $E_F = E_{100} + \hbar\omega_k$, where $\hbar\omega_k$ is the energy of the emitted photon.

a) The quantized electromagnetic field is

$$\mathbf{A}(\mathbf{r}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}\boldsymbol{\lambda}} \left(A_{\mathbf{k}\boldsymbol{\lambda}} \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}} \boldsymbol{\lambda} + A^\dagger_{\mathbf{k}\boldsymbol{\lambda}} \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{r}} \boldsymbol{\lambda}^* \right) .$$

Let us evaluate $\langle F | \mathbf{A}\cdot\mathbf{p} | I \rangle$:

$$\langle F | \mathbf{A}\cdot\mathbf{p} | I \rangle = \langle 100 | \left(\langle \mathbf{k}\boldsymbol{\lambda} | \mathbf{A} | 0 \rangle \cdot \mathbf{p} \right) | nlm \rangle .$$

In computing $\langle \mathbf{k}\boldsymbol{\lambda} | \mathbf{A} | 0 \rangle$, we note that the only non-zero contribution is given by the term $A^\dagger_{\mathbf{k}\boldsymbol{\lambda}}$ and that

$$\begin{aligned} A^\dagger_{\mathbf{k}\boldsymbol{\lambda}} |0\rangle &= \left(\frac{2\pi\hbar c^2}{\omega_k} \right)^{1/2} a^\dagger_{\mathbf{k}\boldsymbol{\lambda}} |0\rangle \\ &= \left(\frac{2\pi\hbar c^2}{\omega_k} \right)^{1/2} |\mathbf{k}\boldsymbol{\lambda}\rangle . \end{aligned}$$

Therefore, we have

$$\langle \mathbf{k}\boldsymbol{\lambda} | \mathbf{A} | 0 \rangle = \left(\frac{2\pi\hbar c^2}{V\omega_k} \right)^{1/2} \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{r}} \boldsymbol{\lambda}^* .$$

We can rewrite $\langle F | \mathbf{A}\cdot\mathbf{p} | I \rangle$ as:

$$\langle F | \mathbf{A}\cdot\mathbf{p} | I \rangle = \left(\frac{2\pi\hbar c^2}{V\omega_k} \right)^{1/2} \boldsymbol{\lambda}^* \cdot \langle 100 | \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{r}} \mathbf{p} | nlm \rangle .$$

In the dipole approximation, we have

$$\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{r}} \approx \mathcal{I} \qquad \Rightarrow \qquad \langle F | \mathbf{A}\cdot\mathbf{p} | I \rangle = \left(\frac{2\pi\hbar c^2}{V\omega_k} \right)^{1/2} \boldsymbol{\lambda}^* \cdot \langle 100 | \mathbf{p} | nlm \rangle .$$

To compute $\langle 100|\, \mathbf{p}\, |nlm\rangle$, use that $\mathbf{p} = \dfrac{m}{i\hbar}[\mathbf{r}, H_0]$ with H_0 the Hamiltonian of the hydrogen atom as shown here:

$$\begin{aligned}\frac{m}{i\hbar}[\mathbf{r}, H_0] &= \frac{m}{i\hbar}[\mathbf{r}, \frac{\mathbf{p}^2}{2m} + V(\mathbf{r})] \\ &= \frac{m}{i\hbar}[\mathbf{r}, \frac{\mathbf{p}^2}{2m}] \\ &= \frac{1}{i2\hbar}\left([\mathbf{r},\mathbf{p}]\mathbf{p} + \mathbf{p}[\mathbf{r},\mathbf{p}]\right) = \mathbf{p}\,,\end{aligned}$$

where we used $[A, BC] = [A, B]C + B[A, C]$. Therefore, in the dipole approximation, we have

$$\begin{aligned}\langle F|\, \mathbf{A}\cdot\mathbf{p}\, |I\rangle &= \left(\frac{2\pi\hbar c^2}{V\omega_k}\right)^{1/2} \boldsymbol{\lambda}^*\cdot \langle 100|\, \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{r}}\mathbf{p}\, |nlm\rangle \\ &\approx \left(\frac{2\pi\hbar c^2}{V\omega_k}\right)^{1/2} \boldsymbol{\lambda}^*\cdot \langle 100|\, \mathbf{p}\, |nlm\rangle \\ &= \left(\frac{2\pi\hbar c^2}{V\omega_k}\right)^{1/2} \boldsymbol{\lambda}^*\cdot \frac{m}{i\hbar}\langle 100|\, [\mathbf{r}, H_0]\, |nlm\rangle \\ &= \left(\frac{2\pi\hbar c^2}{V\omega_k}\right)^{1/2} \boldsymbol{\lambda}^*\cdot \frac{m}{i\hbar}(E_{lmn} - E_{100})\,\langle 100|\, \mathbf{r}\, |nlm\rangle \\ &= \left(\frac{2\pi\hbar c^2}{V\omega_k}\right)^{1/2} \frac{m}{i\hbar}\Delta E\, \boldsymbol{\lambda}^*\cdot \langle 100|\, \mathbf{r}\, |nlm\rangle\ ,\end{aligned}$$

where $\Delta E = E_{nlm} - E_{100}$.

b) Using Fermi's golden rule, we have that the transition rate from $|I\rangle$ to $|F\rangle$ to first order is given by

$$W_{I\to F} = \frac{2\pi}{\hbar}\left|\langle F|\, H_{\text{int}}\, |I\rangle\right|^2 \delta(E_F - E_I)\,,$$

The perturbation is the interaction between the atom and the field:

$$H_{\text{int}} = -\frac{e}{mc}\mathbf{A}\cdot\mathbf{p} + \frac{e^2}{2mc^2}A^2\,,$$

where e is the charge of the electron ($e < 0$) and we work in the radiation gauge ($\nabla\cdot\mathbf{A} = 0$). We can omit the A^2 term since $\langle F|\, A^2\, |I\rangle = 0$: A^2 to first order can create two photons or delete two photons or create one photon and at the same time delete one photon.

¿From the result in a), we have:

$$\left|\langle F| - \frac{e}{mc}\mathbf{A}\cdot\mathbf{p}\, |I\rangle\right|^2 = \frac{e^2}{m^2c^2}\frac{2\pi\hbar c^2}{V\omega_k}\frac{m^2}{\hbar^2}\Delta E^2\left|\langle 100|\, \boldsymbol{\lambda}^*\cdot\mathbf{r}\, |nlm\rangle\right|^2$$

The transition rate is therefore given by

$$W_{I\to F} = (2\pi)^2 \frac{1}{V\omega_k} \frac{e^2}{\hbar^2} \Delta E^2 \left|\langle 100| \boldsymbol{\lambda}^* \cdot \mathbf{r} |nlm\rangle\right|^2 \delta(\hbar\omega_k - \Delta E)\,,$$

c) We now expand the product $\boldsymbol{\lambda}^* \cdot \mathbf{r}$ in Y_{lm} by using

$$\begin{cases} x = \frac{1}{2}\sqrt{\frac{8\pi}{3}} r\,(-Y_{11} + Y_{1-1}) \\ y = \frac{i}{2}\sqrt{\frac{8\pi}{3}} r\,(Y_{11} + Y_{1-1}) \\ z = \sqrt{\frac{4\pi}{3}} r\, Y_{10}\,, \end{cases}$$

where $r = |\mathbf{r}|$. We have

$$\begin{aligned} \boldsymbol{\lambda}^* \cdot \mathbf{r} &= \sqrt{\frac{4\pi}{3}} \left[\frac{1}{\sqrt{2}}(-\lambda_x^* + i\lambda_y^*)\, Y_{11} + \frac{1}{\sqrt{2}}(\lambda_x^* + i\lambda_y^*)\, Y_{1-1} + \lambda_z^*\, Y_{10} \right] \\ &= \sqrt{\frac{4\pi}{3}} \left(-\lambda_{-1}^*\, Y_{11} - \lambda_1^*\, Y_{1-1} + \lambda_0^*\, Y_{10} \right)\,, \end{aligned}$$

where we defined

$$\begin{cases} \lambda_1^* = \frac{-1}{\sqrt{2}}(\lambda_x^* + i\lambda_y^*) \\ \lambda_{-1}^* = \frac{1}{\sqrt{2}}(-\lambda_x^* + i\lambda_y^*) \\ \lambda_0^* = \lambda_z^*\,. \end{cases}$$

d) The expectation value of $\boldsymbol{\lambda}^* \cdot \mathbf{r}$ becomes

$$\begin{aligned} \langle 100| \boldsymbol{\lambda}^* \cdot \mathbf{r} |21m\rangle &= \int \mathrm{d}\mathbf{r}\, \langle 100|\mathbf{r}\rangle\; \boldsymbol{\lambda}^* \cdot \mathbf{r}\; \langle \mathbf{r}|21m\rangle \\ &= \int \mathrm{d}\mathbf{r}\, \Psi_{100}^*(\mathbf{r})\, \boldsymbol{\lambda}^* \cdot \mathbf{r}\, \Psi_{21m}(\mathbf{r}) \\ &= \int \mathrm{d}\mathbf{r}\, R_{10}(r) \frac{1}{\sqrt{4\pi}} \boldsymbol{\lambda}^* \cdot \mathbf{r}\, R_{21}(r) Y_{1m}(\Omega) \\ &= \int \mathrm{d}\mathbf{r}\, R_{10}(r) \frac{1}{\sqrt{4\pi}} \sqrt{\frac{4\pi}{3}} r \left(-\lambda_{-1}^*\, Y_{11} - \lambda_1^*\, Y_{1-1} + \lambda_0^*\, Y_{10} \right) R_{21}(r) Y_{1m}(\Omega) \\ &= \frac{1}{\sqrt{3}} \int_0^\infty \mathrm{d}r\, R_{10}(r) R_{21}(r)\, r^3 \int \mathrm{d}\Omega \left(-\lambda_{-1}^*\, Y_{11} - \lambda_1^*\, Y_{1-1} + \lambda_0^*\, Y_{10} \right) Y_{1m}(\Omega) \\ &= \frac{1}{\sqrt{3}} \int_0^\infty \mathrm{d}r\, R_{10}(r) R_{21}(r)\, r^3 \int \mathrm{d}\Omega \left(\lambda_{-1}^*\, Y_{1-1}^* + \lambda_1^*\, Y_{11}^* + \lambda_0^*\, Y_{10}^* \right) Y_{1m}(\Omega) \\ &= \frac{1}{\sqrt{3}} \int_0^\infty \mathrm{d}r\, R_{10}(r) R_{21}(r)\, r^3 \left(\lambda_1^*\, \delta_{m,1} + \lambda_{-1}^*\, \delta_{m,-1} + \lambda_0^*\, \delta_{m,0} \right)\,, \end{aligned}$$

where we used $\Psi_{nlm}(\mathbf{r}) = R_{nl}(r)Y_{lm}(\Omega)$, $\quad Y_{00} = 1/\sqrt{4\pi}$, $\quad Y_{l,-m}(\Omega) = (-1)^m\, Y^*_{l,m}(\Omega)$, and the orthogonality relation $\int \mathrm{d}\Omega\, Y^*_{l',m'}(\Omega)Y_{l,m}(\Omega) = \delta_{ll'}\delta_{mm'}$.

e) The transition rate can be rewritten as

$$W_{I\to F} = (2\pi)^2 \frac{1}{V\omega_k}\frac{e^2}{\hbar^2}\Delta E^2 \frac{1}{3}\left|\int_0^\infty \mathrm{d}r\, R_{10}(r)R_{21}(r)\, r^3\right|^2 \left|\lambda_1^*\,\delta_{m,1} + \lambda_{-1}^*\,\delta_{m,-1} + \lambda_0^*\,\delta_{m,0}\right|^2 \delta(\hbar\omega_k - \Delta E)\,.$$

By averaging the transition rate over all three initial states $m = 0, \pm 1$, we obtain

$$\begin{aligned}
W_{I\to F} &= \frac{1}{3}\sum_{m=0,\pm1} W_{I_m\to F} \\
&= (2\pi)^2 \frac{1}{V\omega_k}\frac{e^2}{\hbar^2}\Delta E^2 \frac{1}{9}\left|\int_0^\infty \mathrm{d}r\, R_{10}(r)R_{21}(r)\, r^3\right|^2 \left(|\lambda_1|^2 + |\lambda_{-1}|^2 + |\lambda_0|^2\right)\delta(\hbar\omega_k - \Delta E) \\
&= (2\pi)^2 \frac{1}{V\omega_k}\Delta E^2 \frac{2^{15}}{3^{11}}\frac{\hbar^2}{m^2 e^2}\delta(\hbar\omega_k - \Delta E)\,,
\end{aligned}$$

using that the polarization vector is normalized $|\boldsymbol{\lambda}|^2 = 1$ and that

$$\int_0^\infty \mathrm{d}r\, R_{10}(r)R_{21}(r)\, r^3 = \sqrt{\frac{3}{2}\frac{2^8}{3^5}\frac{\hbar^2}{me^2}}\,.$$

f) If we sum over all possible photon momenta and the two polarizations, we obtain

$$\begin{aligned}
2\int \frac{V}{(2\pi)^3}k^2\,\frac{1}{\omega_k}\,\delta(\hbar\omega_k - \Delta E)\,\mathrm{d}k\,\mathrm{d}\Omega &= 2\int \frac{V}{(2\pi)^3}\frac{1}{c^3}\omega_k^2\,\frac{1}{\omega_k}\,\delta(\hbar\omega_k - \Delta E)\,\mathrm{d}\omega_k\,\mathrm{d}\Omega \\
&= 2\frac{V}{(2\pi)^3}\frac{1}{c^3}\int \mathrm{d}\Omega \int \mathrm{d}\omega_k\omega_k\,\delta(\hbar\omega_k - \Delta E) \\
&= 2\frac{V}{(2\pi)^3}\frac{1}{c^3}4\pi \int \mathrm{d}\omega_k\omega_k\,\delta(\hbar\omega_k - \Delta E) \\
&= \frac{V}{\pi^2}\frac{1}{c^3}\frac{1}{\hbar^2}\,\Delta E\,.
\end{aligned}$$

Therefore, the transition rate from the $2p$ state to the ground state emitting a photon is given by

$$\begin{aligned}
W_{I\to \mathrm{all}} &= \frac{(2\pi)^2}{V}\frac{2^{15}}{3^{11}}\frac{\hbar^2}{m^2e^2}\Delta E^2 \frac{V}{\pi^2}\frac{1}{c^3}\frac{1}{\hbar^2}\,\Delta E \\
&= \frac{2^{17}}{3^{11}}\frac{1}{c^3m^2e^2}\Delta E^3 \\
&= \frac{2^{17}}{3^{11}}\frac{1}{c^3m^2e^2}\frac{3^3}{2^9}\frac{m^3e^{12}}{\hbar^6}
\end{aligned}$$

$$= \frac{2^8}{3^8} \frac{e^{10} m}{c^3 \hbar^6}$$

$$= \left(\frac{2}{3}\right)^8 \alpha^5 \frac{mc^2}{\hbar} \,,$$

where we used that the energies of the hydrogen atom are $E_{nlm} = -\frac{e^2}{2n^2 a_0}$ with $a_0 = \frac{\hbar^2}{me^2}$ so that $\Delta E = E_{21m} - E_{100} = \frac{3}{8} \frac{e^2}{a_0}$.

Finally, by substituting $\alpha = \frac{e^2}{\hbar c} = \frac{1}{137}$ and $\frac{mc^2}{\hbar} = 7.78 \times 10^{20}$ 1/sec, we obtain

$$W_{I \to \text{all}} \approx 0.6 \times 10^9 \ \text{1/sec} \qquad \Rightarrow \qquad T \approx 1.6 \times 10^{-9} \quad \text{sec} \,,$$

where we used that the lifetime for the hydrogen atom in the $2p$ state is $T = 1/W_{I \to \text{all}}$.