Solutionset 5.

Excercise 1

(a) Let us first determine the operators a and $a^\dagger$ in the Heisenberg picture. Our Hamiltonian is given by $\hat{\mathcal{H}} = \hbar\omega(a^\dagger a + \frac{1}{2})$.

$$\begin{aligned}
\frac{d}{dt}a &= \frac{i}{\hbar}[H, a] = i\omega[a^\dagger a, a] = -i\omega a \Rightarrow a(t) = e^{-i\omega t}a \\
\frac{d}{dt}a^\dagger &= \frac{i}{\hbar}[H, a^\dagger] = i\omega[a^\dagger a, a^\dagger] = i\omega a^\dagger \Rightarrow a^\dagger(t) = e^{i\omega t}a^\dagger
\end{aligned}$$

Therefore the time evolution of $\hat{x}$ and $\hat{p}$ is

$$\begin{aligned}
\hat{x}(t) &= \sqrt{\frac{\hbar}{2\omega m}}(a(t) + a^\dagger(t)) = \sqrt{\frac{\hbar}{2\omega m}}(ae^{-i\omega t} + a^\dagger e^{i\omega t}), \\
\hat{p}(t) &= -i\sqrt{\frac{\hbar\omega m}{2}}(a(t) - a^\dagger(t)) = -i\sqrt{\frac{\hbar\omega m}{2}}(ae^{-i\omega t} - a^\dagger e^{i\omega t}).
\end{aligned}$$

Using $a|\lambda\rangle = \lambda|\lambda\rangle$ and $\langle\lambda|a^\dagger = \langle\lambda|\lambda^*$ it is straightforward to calculate

$$\begin{aligned}
\langle\hat{x}(t)\rangle &= \sqrt{\frac{\hbar}{2\omega m}}(\lambda e^{-i\omega t} + \lambda^* e^{i\omega t}), \\
\langle\hat{p}(t)\rangle &= -i\sqrt{\frac{\hbar\omega m}{2}}(\lambda e^{-i\omega t} - \lambda^* e^{i\omega t}), \\
\langle\hat{x}^2(t)\rangle &= \frac{\hbar}{2\omega m}(\lambda^2 e^{-2i\omega t} + 1 + 2\lambda\lambda^* + (\lambda^*)^2 e^{2i\omega t}), \\
\langle\hat{p}^2(t)\rangle &= -\frac{\hbar\omega m}{2}(\lambda^2 e^{-2i\omega t} - 1 - 2\lambda\lambda^* + (\lambda^*)^2 e^{2i\omega t}).
\end{aligned} \tag{1}$$

From that we obtain

$$\begin{aligned}
\langle(\Delta\hat{x}(t))^2\rangle &= \langle\hat{x}^2(t)\rangle - \langle\hat{x}(t)\rangle^2 = \frac{\hbar}{2\omega m}, \\
\langle(\Delta\hat{p}(t))^2\rangle &= \langle\hat{p}^2(t)\rangle - \langle\hat{p}(t)\rangle^2 = \frac{\hbar\omega m}{2}.
\end{aligned}$$

Therefore the uncertainty is minimal for all times, $\langle(\Delta\hat{x}(t))^2\rangle\langle(\Delta\hat{p}(t))^2\rangle = \frac{\hbar^2}{4}$.

(b) The direct calculation of the energy of the coherent state $|\lambda\rangle$ gives us

$$E_\lambda = \langle\lambda|\hat{\mathcal{H}}|\lambda\rangle = \hbar\omega(|\lambda|^2 + \frac{1}{2}).$$

Of course, we get the same results for $E = \frac{\langle\hat{p}^2(t)\rangle}{2m} + \frac{m\omega^2\langle\hat{x}^2(t)\rangle}{2}$ using the results from (a).

(c) We have seen in a previous homework that we obtain a coherent state $|\lambda\rangle$ if we apply the translation operator $\exp(-i\hat{p}l/\hbar)$ to the ground state $|0\rangle$. We have found $\lambda = l\sqrt{\frac{m\omega}{2\hbar}}$. Let us now plug this value into the results obtained in (a) and (b):

$$\begin{aligned}\langle\hat{x}(t)\rangle &= l\cos(\omega t),\\ \langle\hat{p}(t)\rangle &= -lm\omega\sin(\omega t),\\ E &= \frac{1}{2}m\omega^2 l^2 + \frac{\hbar\omega}{2}.\end{aligned}$$

Therefore these expectation values correspond perfectly to a classical harmonic oscillator.

Excercise 2

(a) The first part of the Hamiltonian tells us that the creation of the electron at some fixed position costs an energy ϵ, the second part is just the standard phonon part (already in a diagonal form). The last part is the electron-phonon coupling. The elecron interacts with the ions forming the lattice. These ions displace from their equilibrium positions. The expansion of the interaction between the electron and the lattice ions gives us to linear order a coupling to the displacement of the lattice and therefore a coupling to $a + a^\dagger$.

(b) The prove of the Baker-Hausdorff relation is quite easy if we use the right trick. We start with the function $f(\lambda) := e^{\lambda S} A e^{-\lambda S}$. The derivative of this functions is given by

$$f'(\lambda) = S e^{\lambda S} A e^{-\lambda S} - e^{\lambda S} A e^{-\lambda S} S = [S, f(\lambda)].$$

If we insert the taylor expansion $f(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) \lambda^n$ into this relation, we obtain the recursion relations

$$[S, f^{(n)}(0)] = f^{(n+1)}(0), \quad f^{(0)}(0) = S.$$

Therefore we have

$$f(\lambda) = S + \lambda [S, A] + \frac{\lambda^2}{2!}[S,[S,A]] + \frac{\lambda^3}{3!}[S,[S,[S,A]]] + \dots .$$

The desired result follows for $\lambda = 1$.

(c) Now we take $S = -c^\dagger c \sum_k \frac{M_k}{\epsilon_k}(a_k^\dagger - a_k) =: -c^\dagger c B$. Since $[S, c] = -B[c^\dagger c, c] = Bc$ we obtain

$$\bar{c} = e^S c e^{-S} = c \exp(B) = cX,$$

and from that directly $\bar{c}^\dagger = c^\dagger X^\dagger$. To get the transformed Bose operators we start to calculate

$$[S, a_k] = -c^\dagger c [\sum_q \frac{M_q}{\epsilon_q}(a_q^\dagger - a_q), a_k] = -c^\dagger c \frac{M_k}{\epsilon_k}[a_k^\dagger, a_k] = \frac{M_k}{\epsilon_k} c^\dagger c.$$

From that we see that $[S,[S,a_k]] = [S,[S,[S,a_k]]] = \ldots = 0$ and hence

$$\bar{a}_k = a_k + [S, a_k] = a_k + \frac{M_k}{\epsilon_k} c^\dagger c.$$

Hermitian conjugation gives us

$$\bar{a}_k^\dagger = a_k + [S, a_k] = a_k^\dagger + \frac{M_k}{\epsilon_k} c^\dagger c.$$

(d) We use these transformations (the inverse ones) to rewrite the Hamiltonian:

$$\begin{aligned}
\hat{\mathcal{H}} &= \epsilon c^\dagger c + \sum_k \epsilon_k a_k^\dagger a_k + c^\dagger c \sum_k M_k (a_k + a_k^\dagger) \\
&= \epsilon \bar{c}^\dagger \bar{c} + \sum_k \epsilon_k (\bar{a}_k^\dagger - \frac{M_k}{\epsilon_k} \bar{c}^\dagger \bar{c})(\bar{a}_k - \frac{M_k}{\epsilon_k} \bar{c}^\dagger \bar{c}) + \bar{c}^\dagger \bar{c} \sum_k M_k (\bar{a}_k^\dagger + \bar{a}_k - 2\frac{M_k}{\epsilon_k} \bar{c}^\dagger \bar{c}) \\
&= (\epsilon - \sum_k \frac{M_k^2}{\epsilon_k}) \bar{c}^\dagger \bar{c} + \sum_k \epsilon_k \bar{a}_k^\dagger \bar{a}_k .
\end{aligned}$$

The electron is dressed by phonons. Or in other words, the lattice locally deforms in the vicinity of the fixed electron. This leads to a reduction of the electronic energy due to screening by $\Delta = \sum_k \frac{M_k^2}{\epsilon_k}$.

Excercise 3

(a) Making the transformation

$$\begin{aligned} b &= ua + va^\dagger, \\ b^\dagger &= ua^\dagger + va, \end{aligned}$$

where u, v are assumed to be real here, we find that

$$[b, b^\dagger] = [ua + va^\dagger, ua^\dagger + va] + u^2[a, a^\dagger] + v^2[a^\dagger, a] = u^2 - v^2,$$

and $[b, b] = [b^\dagger, b^\dagger] = 0$ trivially. If $u^2 - v^2 = 1$, then $[b, b^\dagger] = 1$ and the transformation is canonical.

(b) Let us assume that the hamiltonian can be diagonalized in the form

$$\hat{\mathcal{H}} = \tilde{\omega}(b^\dagger b + \frac{1}{2}).$$

Substituting in the above transformation, we find that

$$\hat{\mathcal{H}} = \omega(a^\dagger a + \frac{1}{2}) + \frac{1}{2}\Delta(a^\dagger a^\dagger + aa),$$

where

$$\omega = \tilde{\omega}(u^2 + v^2), \quad \Delta = 2\tilde{\omega}uv.$$

Squaring both terms and substracting, we find

$$\omega^2 - \Delta^2 = \tilde{\omega}^2(u^2 - v^2)^2 = \tilde{\omega}^2$$

so that $\tilde{\omega} = \sqrt{\omega^2 - \Delta^2}$. By substituting $v^2 = u^2 - 1$ into $\omega = \tilde{\omega}(u^2 + v^2)$, we obtain

$$\begin{aligned} u^2 &= \frac{1}{2}(1 + \frac{\omega}{\sqrt{\omega^2 - \Delta^2}}), \\ v^2 &= -\frac{1}{2}(1 - \frac{\omega}{\sqrt{\omega^2 - \Delta^2}}). \end{aligned}$$

When $\Delta = \omega$, the frequency of oscillation goes to zero. You might perhaps have spotted that if you write $a = (x + ip)/\sqrt{2}$, then $(a^\dagger)^2 + a^2 = (x^2 - p^2)$, so that when $\Delta = \omega$, the Hamiltonian takes the form $\hat{\mathcal{H}} = \omega x^2$, i.e. the mass of the particle has become infinite, and hence the freqeuncy of oscillation vanishes.

(c) The ground state is given by $|0_b\rangle$, $E_0 = \hat{\mathcal{H}}|0_b\rangle = \tilde{\omega}/2$. For this state we have $b|0_b\rangle = 0$ ($\langle 0_b|b^\dagger = 0$). The Bogoliubov transformation is inversed by

$$\begin{aligned} a^\dagger &= ub - vb^\dagger, \\ a &= ub^\dagger - vb. \end{aligned}$$

Therefore we directly obtain

$$\langle a\rangle_b = \langle 0_b|a|0_b\rangle = \langle 0_b|ub^\dagger - vb|0_b\rangle = 0.$$

Form that we can easily determine

$$\langle a\rangle_b = \langle a^\dagger\rangle_b^* = 0.$$

Using the inverse transformation we can also easily calculate the expectation value of the number operator $\hat{n}_a = a^\dagger a$:

$$\langle a^\dagger a\rangle_b = \langle v^2 + (u^2 + v^2)b^\dagger b - uv(bb + b^\dagger b^\dagger)\rangle_b = v^2 = -\frac{1}{2}(1 - \frac{\omega}{\sqrt{\omega^2 - \Delta^2}}).$$

For $\Delta = 0$ we have $n_a = 0$, what is clear since the Hamiltonian is already diagonal. In the limit $\Delta \to \omega$ we have $n_a \to \infty$ (Bose condensation).

Excercise 4

(a) We have to show that $\{c_1, c_1^\dagger\} = \{c_2, c_2^\dagger\} = 1$ and also $\{c_1, c_2\} = \{c_1, c_2^\dagger\} = 0$. Substituting for c_1 and c_2, we obtain

$$
\begin{aligned}
\{c_1, c_2\} &= \{ua_1 + va_2^\dagger, -va_1^\dagger + ua_2\} = -uv\{a_1, a_1^\dagger\} + vu\{a_2^\dagger, a_2\} = 0, \\
\{c_1, c_2^\dagger\} &= \{ua_1 + va_2^\dagger, -v^* a_1 + u^* a_2^\dagger\} = 0, \\
\{c_1, c_1^\dagger\} &= \{ua_1 + va_2^\dagger, u^* a_1^\dagger + v^* a_2\} = uu^*\{a_1, a_1^\dagger\} + vv^*\{a_2^\dagger, a_2\} = |u|^2 + |v|^2 = 1, \\
\{c_2, c_2^\dagger\} &= \{-va_1^\dagger + ua_2, -v^* a_1 + u^* a_2^\dagger\} = vv^*\{a_1^\dagger, a_1\} + uu^*\{a_2, a_2^\dagger\} = |u|^2 + |v|^2 = 1,
\end{aligned}
$$

(b) Consider $\hat{\mathcal{H}} = \omega(c_1^\dagger c_1 - c_2 c_2^\dagger)$, then if

$$
\begin{pmatrix} c_1 \\ c_2^\dagger \end{pmatrix} = \begin{pmatrix} u & v \\ -v^* & u^* \end{pmatrix} \begin{pmatrix} a_1 \\ a_2^\dagger \end{pmatrix} =: U \begin{pmatrix} a_1 \\ a_2^\dagger \end{pmatrix},
$$

where we note U is a unitary transformation, we may rewrite $\hat{\mathcal{H}}$ as

$$
\hat{\mathcal{H}} = (c_1^\dagger, c_2) \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix} \begin{pmatrix} c_1 \\ c_2^\dagger \end{pmatrix}
$$

so that using the transformation

$$
\begin{aligned}
\hat{\mathcal{H}} &= (a_1^\dagger, a_2) U^\dagger \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix} U \begin{pmatrix} a_1 \\ a_2^\dagger \end{pmatrix} \\
&= (a_1^\dagger, a_2) \begin{pmatrix} \epsilon & \Delta \\ \Delta^* & -\epsilon \end{pmatrix} \begin{pmatrix} a_1 \\ a_2^\dagger \end{pmatrix} \\
&= \epsilon(a_1^\dagger a_1 - a_2 a_2^\dagger) + (\Delta a_1^\dagger a_2^\dagger + \text{h.c.})
\end{aligned}
$$

where

$$
\begin{aligned}
\epsilon &= \omega(|u|^2 - |v|^2), \\
\Delta &= 2\omega u^* v.
\end{aligned}
$$

Squaring both expressions and adding the results, we obtain $\omega = (\epsilon^2 + \Delta^2)^{\frac{1}{2}}$ and

$$
|u|^2 = \frac{1}{2}\left(1 + \frac{\epsilon}{\omega}\right), \quad |v|^2 = \frac{1}{2}\left(1 - \frac{\epsilon}{\omega}\right).
$$

(c) The ground-state is annihilated by both c_1 and c_2, so that if $\hat{\mathcal{H}} = \omega(c_1^\dagger c_1 + c_2^\dagger c_2 - 1)$, the ground-state energy is $E_0 = -\omega = -(\epsilon^2 + \Delta^2)^{\frac{1}{2}}$.

To transform the ground state $|0,0\rangle_c$ to the a-basis $\{|n_1,n_2\rangle_a\}$ we start from

$$|0,0\rangle_c = \alpha_{00}|0,0\rangle_a + \alpha_{01}|0,1\rangle_a + \alpha_{10}|1,0\rangle_a + \alpha_{11}|1,1\rangle_a$$

and use the fact that applying $c_1 = ua_1 + va_2^\dagger$ or $c_2 = -va_1^\dagger + ua_2$ to this states gives us zero.

$$\begin{aligned} 0 &= c_1|0,0\rangle_c = (ua_1 + va_2^\dagger)(\alpha_{00}|0,0\rangle_a + \alpha_{01}|0,1\rangle_a + \alpha_{10}|1,0\rangle_a + \alpha_{11}|1,1\rangle_a) \\ &= u\alpha_{10}|0,0\rangle_a + (u\alpha_{11} + v\alpha_{00})|0,1\rangle_a - v\alpha_{10}|1,1\rangle_a \\ 0 &= c_2|0,0\rangle_c = (ua_2 - va_1^\dagger)(\alpha_{00}|0,0\rangle_a + \alpha_{01}|0,1\rangle_a + \alpha_{10}|1,0\rangle_a + \alpha_{11}|1,1\rangle_a) \\ &= u\alpha_{01}|0,0\rangle_a - (u\alpha_{11} + v\alpha_{00})|1,0\rangle_a - v\alpha_{01}|1,1\rangle_a \end{aligned}$$

From that we immediately get that $\alpha_{01} = \alpha_{10} =$ and $u\alpha_{11} + v\alpha_{00} = 0$. Therefore we can write the normalized ground state as

$$|0,0\rangle_c = u|0,0\rangle_a - v|1,1\rangle_a,$$

where the Cooper pair contribution is given by $|1,1\rangle_a$. Its weight is given by

$$|\ {}_c\langle 0,0|1,1\rangle_a|^2 = |v|^2 = \frac{1}{2}\left(1 - \frac{\epsilon}{\sqrt{\epsilon^2 + \Delta^2}}\right).$$