Problem 1

(1.) (Sakurai 1-12)

a) From last homework: $|\hat{n},+\rangle = \cos\frac{\beta}{2}|+\rangle + e^{i\alpha}\sin\frac{\beta}{2}|-\rangle$

Since $\hat{n}$ lies in x-z plane and makes angle γ with positive z-axis,

$$|\hat{n};+\rangle = \cos\frac{\gamma}{2}|+\rangle + \sin\frac{\gamma}{2}|-\rangle$$

then,

$$\langle\hat{x};+|\hat{n};+\rangle = \left[\frac{\langle+| + \langle-|}{\sqrt{2}}\right]\left[\cos\frac{\gamma}{2}|+\rangle + \sin\frac{\gamma}{2}|-\rangle\right] = \frac{1}{\sqrt{2}}\left[\cos\frac{\gamma}{2} + \sin\frac{\gamma}{2}\right]$$

$$\Rightarrow \text{probability} \equiv |\langle\hat{x};+|\hat{n};+\rangle|^2 = \frac{1+\sin\gamma}{2}$$

b) $\langle(S_x - \langle S_x\rangle)^2\rangle = \langle S_x^2\rangle - \langle S_x\rangle^2$

Use 1.4.18 in Sakurai to calculate above expectation values:

$$\langle S_x\rangle = \langle\hat{n};+|S_x|\hat{n};+\rangle = \frac{\hbar}{2}\left[\cos\frac{\gamma}{2}\langle+| + \sin\frac{\gamma}{2}\langle-|\right]\left[\cos\frac{\gamma}{2}|-\rangle + \sin\frac{\gamma}{2}|+\rangle\right]$$

$$= \frac{\hbar}{2}\sin\gamma$$

$$\langle S_x^2\rangle = \frac{\hbar}{2}\langle\hat{n};+|S_x|\left(\cos\frac{\gamma}{2}|-\rangle + \sin\frac{\gamma}{2}|+\rangle\right) = \frac{\hbar^2}{4}$$

$$\Rightarrow \langle(S_x - \langle S_x\rangle)^2\rangle = \frac{\hbar^2}{4}(1-\sin^2\gamma) = \frac{\hbar^2}{4}\cos^2\gamma$$

γ	dispersion
0	$\hbar^2/4$
$\pi/2$	0
π	$\hbar^2/4$

Problem 2

(2.) (Sakurai 1-20)

From the symmetry of $\langle(\Delta S_x)^2\rangle\langle(\Delta S_y)^2\rangle$ it is obvious that the product will be maximized if $\langle(\Delta S_x)^2\rangle = \langle(\Delta S_y)^2\rangle = \text{const.}$ Therefore, we need to find a state which treats S_x and

S_y the same way. The state which has no preference to S_x over S_y is eigenstate of S_z. We find that $\pm|+\rangle$ and $\pm|-\rangle$ will maximize the above product.

From previous problem, $\langle (\Delta S_x)^2 \rangle = \frac{\hbar^2}{4}$ with $\theta = 0$ or π.

$\Rightarrow \langle (\Delta S_x)^2 \rangle \langle (\Delta S_y)^2 \rangle = \frac{\hbar^4}{16}$

In order to calculate the commutator, we use the following:

$$[S_x, S_y] = i\hbar S_z$$

$\Rightarrow \langle [S_x, S_y] \rangle = i\hbar \langle S_z \rangle = i\frac{\hbar^2}{2}$, or $|\langle [S_x, S_y] \rangle|^2 = \frac{\hbar^4}{4}$

Finally we get the equality:

$$\langle (\Delta S_x)^2 \rangle \langle (\Delta S_y)^2 \rangle = \frac{1}{4} |\langle [S_x, S_y] \rangle|^2.$$ So, the uncertainty relationship for S_x and S_y is not violated.

Problem 3

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3. a) $\langle (\Delta S_x)^2 \rangle = 0$ since we have eigenstate of S_z. Therefore, left side of 1.4.53 $= 0$, and $\langle [S_z, S_x] \rangle = i\hbar \langle S_y \rangle = 0$. So, right side of 1.4.53 $= 0$ and the equation is correct.

b) $\langle [\Delta S_z, \Delta S_x] \rangle = \langle \Delta S_z \Delta S_x + \Delta S_x \Delta S_z \rangle =$

$$= \langle (S_z - \langle S_z \rangle)(S_x - \langle S_x \rangle) + (S_x - \langle S_x \rangle)(S_z - \langle S_z \rangle) \rangle$$
$$= \langle (S_z - \tfrac{\hbar}{2}) S_x + S_x (S_z - \tfrac{\hbar}{2}) \rangle = \langle S_z S_x + S_x S_z - \hbar S_x \rangle$$
$$= 0$$

Since the anti-commutator vanishes, the uncertainity relation still holds.

Problem 4

Let us recall the defining properties of the (generalized) function $\delta(x)$:

$$\delta(x) = 0 \quad \text{for } x \neq 0 \quad , \tag{1}$$
$$\int \delta(x) dx = 1 \quad , \tag{2}$$
$$\int f(x)\delta(x) dx = f(0) \quad . \tag{3}$$

Formulas (2) and (3) are valid for any integration domain containing the point $x = 0$. Withouth loss of generality we will assume it to be $(-\infty, +\infty)$. Equipped with these formulas we can study the general expression

$$I \equiv \int_{-\infty}^{+\infty} f(x)\, \delta\left(g(x)\right) dx \quad , \tag{4}$$

where the function $g(x)$ is assumed to have simple zeroes at the locations x_i:

$$g\left(x_i\right) = 0 \ , \quad i = 1, \ldots, n \quad , \tag{5}$$
$$\left(\frac{dg}{dx}\right)_{x=x_i} \neq 0 \quad . \tag{6}$$

Making repeated use of property (1) we can rewrite (4) as follows:

$$I = \sum_{i=1}^{n} \int_{x_i-\epsilon}^{x_i+\epsilon} f(x)\, \delta\left(g(x)\right) dx \quad , \tag{7}$$

where $\epsilon > 0$ is arbitrarily small.

We now change the integration variable in (7):

$$y = g(x) \ , \quad dy = \frac{dg}{dx} dx \quad . \tag{8}$$

We get

$$I = \sum_{i=1}^{n} \int_{g(x_i-\epsilon)}^{g(x_i+\epsilon)} f\left(g^{-1}(y)\right) \delta(y) \frac{dy}{dg/dx} \quad . \tag{9}$$

Notice that, as a consequence of (6), the inverse function g^{-1} is well defined in a neighbourhood of x_i.

Now the prescription (3) tells us to set everywhere $y = 0$, or $x = x_i$. There is one subtlety left, though:

- if $\left(\frac{dg}{dx}\right)_{x=x_i} > 0$, then $g(x_i + \epsilon) > g(x_i - \epsilon)$;
- if $\left(\frac{dg}{dx}\right)_{x=x_i} < 0$, then $g(x_i + \epsilon) < g(x_i - \epsilon)$.

With this we can finally write

$$\int_{-\infty}^{+\infty} f(x)\, \delta\left(g(x)\right) dx = \sum_{i=1}^{n} \left|\frac{dg}{dx}\right|_{x=x_i}^{-1} f(x_i) \quad . \tag{10}$$

With the above general result at hand, we can immediately write the following formulas:

$$\int_{-\infty}^{+\infty} f(x)\delta(ax)dx = \frac{1}{|a|} f(0) \quad , \tag{11}$$

$$\int_{-\infty}^{+\infty} f(x)\delta(x^2 - a^2)dx = \frac{1}{2|a|}\left[f(-a) + f(a)\right] \quad . \tag{12}$$

Let us now address the question about the derivative of the δ-function:

$$I' \equiv \int_{-\infty}^{+\infty} f(x) \frac{d\delta(x)}{dx} dx \quad . \tag{13}$$

Integrating by parts we can write

$$I' = \left[f(x)\delta(x)\right]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \frac{df}{dx} \delta(x) dx \quad . \tag{14}$$

The integrated term vanishes because of (1) and the remaining integral is evaluated by means of (3). So we finally establish the following:

$$\int_{-\infty}^{+\infty} f(x) \frac{d\delta(x)}{dx} dx = -\left(\frac{df}{dx}\right)_{x=0} \quad . \tag{15}$$

Problem ~~6~~ 5

We start with proving that

$$[x, p^n] = in\hbar p^{n-1} \quad . \tag{28}$$

Using the fact that

$$[A, BC] = B[A, C] + [A, B]C \tag{29}$$

we immediately find that (28) holds for $n = 2$:

$$[x, p^2] = p[x, p] + [x, p]p = 2i\hbar p \quad ; \tag{30}$$

we can also write

$$[x, p^n] = p[x, p^{n-1}] + [x, p]p^{n-1} \quad , \tag{31}$$

which, assuming the property (28) holds for n=n-1, implies the assert:

$$[x, p^n] = in\hbar p^{n-1} \quad , \tag{32}$$

thus completing our proof by induction.

We can now calculate any commutator of the form

$$[x, F(p)] \quad , \tag{33}$$

where the operator valued function $F(p)$ is defined through its Taylor expansion:

$$F(p) = \sum_{n=0}^{\infty} a_n p^n \quad . \tag{34}$$

Inserting (34) into (33) we have

$$\begin{aligned} [x, F(p)] &= \sum_{n=0}^{\infty} a_n [x, p^n] \\ &= i\hbar \sum_{n=1}^{\infty} n a_n p^{n-1} \\ &= i\hbar \frac{\partial}{\partial p} \sum_{n=0}^{\infty} a_n p^n \quad , \end{aligned} \tag{35}$$

which is nothing but

$$[x, F(p)] = i\hbar \frac{\partial}{\partial p} F(p) \quad . \tag{36}$$

Let us now consider the action of the operator $F_a(p) = e^{-iap/\hbar}$ on a position eigenstate $|x'\rangle$; we have

$$\begin{aligned} x e^{-iap/\hbar}|x'\rangle &= e^{-iap/\hbar} x |x'\rangle + [x, e^{-iap/\hbar}]|x'\rangle \\ &= x' e^{-iap/\hbar}|x'\rangle + a e^{-iap/\hbar}|x'\rangle \\ &= (x' + a) e^{-iap/\hbar}|x'\rangle \quad , \end{aligned} \tag{37}$$

where (36) has been used to calculate $[x, e^{-iap/\hbar}]$. Eq. (37) tells us that $e^{-iap/\hbar}|x'\rangle$ is still an eigenstate of position corresponding to the eigenvalue $x' + a$ (a restatement of the fact that the momentum is the generator of translations). Since there is no constraint on a, we conclude that the spectrum of the position operator is continuous.

We can now run a specular argument interchanging the role of x and p. Starting from

$$[p, G(x)] = -i\hbar \frac{\partial}{\partial x} G(x) \quad , \tag{38}$$

and considering the action of $G(x) = e^{ikx/\hbar}$ on the momentum eigenstate $|p\rangle$, one finds that $G(x)|p\rangle$ is still a momentum eigenstate corresponding to the eigenvalue $p+k$, which implies that the spectrum of momentum is continuous too.

We can see that the argument above fails in the case of a particle confined in an infinite potential well: there the position spectrum is truncated, $0 \leq x' \leq L$, therefore the translation operator annihilates some of the states.

Indeed, the Hilbert space of square integrable wavefunctions on the segment is spanned by a discrete plane wave basis.

Notice that the boundary condition $\psi(0) = \psi(L) = 0$ makes the problem even more pathological: ordinary plane waves cannot be physical states. This can also be understood on the basis of Heisenberg undeterminacy principle: since Δx is of the order L, the expectation value of the momentum in any state cannot vanish, thereby forbidding momentum eigenstates.

$$\hat{x} F_a(p)|x\rangle = F_a(p)\hat{x}|x\rangle + \overbrace{a F_a(p)}^{[x, F(p)]}|x\rangle$$

$$= (x+a) F(p)|x\rangle$$

$$\Rightarrow F_a(p)|x\rangle = |x+a\rangle;$$ "shift operator"

Problem ~~7~~ 6

(a) Let us start calculating $\langle p \rangle$ using the position space wavefunction (1.7.35):

$$
\begin{aligned}
\langle p \rangle &= \frac{1}{\pi^{1/2} d} \int_{-\infty}^{+\infty} dx' \exp\left(-ikx' - \frac{x'^2}{2d^2}\right)\left(-i\hbar \frac{d}{dx}\right) \exp\left(ikx' - \frac{x'^2}{2d^2}\right) \\
&= \frac{-i\hbar}{\pi^{1/2} d} \int_{-\infty}^{+\infty} dx' \left(ik - \frac{x'}{d^2}\right) \exp\left(-\frac{x'^2}{d^2}\right) \\
&= \hbar k \quad . \qquad (54)
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\left\langle p^2 \right\rangle &= \frac{1}{\pi^{1/2} d} \int_{-\infty}^{+\infty} dx' \exp\left(-ikx' - \frac{x'^2}{2d^2}\right)\left(-\hbar^2 \frac{d^2}{dx^2}\right) \exp\left(ikx' - \frac{x'^2}{2d^2}\right) \\
&= \frac{-\hbar^2}{\pi^{1/2} d} \int_{-\infty}^{+\infty} dx' \left(-k^2 - \frac{1}{d^2} + \frac{x'^2}{d^4}\right) \exp\left(-\frac{x'^2}{d^2}\right) \\
&= \hbar^2 \left(k^2 + \frac{1}{d^2}\right) - \frac{\hbar^2}{2d^2} \\
&= \hbar^2 k^2 + \frac{\hbar^2}{2d^2} \quad , \qquad (55)
\end{aligned}
$$

so that

$$
\left\langle (\Delta p)^2 \right\rangle = \frac{\hbar^2}{2d^2} \quad . \qquad (56)
$$

(b) The same result can be obtained more easily using the momentum space representation (1.7.42):

$$
\begin{aligned}
\langle p \rangle &= \int_{-\infty}^{+\infty} dp' p' |\phi(p')|^2 \\
&= \frac{d}{\hbar \pi^{1/2}} \int_{-\infty}^{+\infty} dp' p' \exp\left(-\frac{d^2 (p' - \hbar k)^2}{2\hbar^2}\right) \\
&= \hbar k \quad ; \qquad (57)
\end{aligned}
$$

$$
\left\langle p^2 \right\rangle = \frac{1}{\pi^{1/2} d} \int_{-\infty}^{+\infty} dp' p'^2 |\phi(p')|^2
$$

$$
\begin{aligned}
&= \frac{d}{\hbar\pi^{1/2}}\int_{-\infty}^{+\infty} dp' p'^2 \exp\left(-\frac{d^2(p'-\hbar k)^2}{2\hbar^2}\right) \\
&= \hbar^2 k^2 + \frac{\hbar^2}{2d^2} \quad , \qquad (58)
\end{aligned}
$$

where we have used the standard integrals quoted on the assignment sheet.

Finally, let us examine the expression

$$
I \equiv \int_{-\infty}^{+\infty} dx' \exp\left(-\frac{x'^2}{2d^2} - ix'\left(\frac{p'}{\hbar} - k\right)\right) \quad . \qquad (59)
$$

(c) Completing the square one can write

$$
I = \exp\left(-\frac{d^2}{2\hbar^2}(p'-\hbar k)^2\right)\int_{-\infty}^{+\infty} dx' \exp\left\{-\frac{1}{2}\left[\frac{x'}{d} + id\left(\frac{p'}{\hbar} - k\right)\right]^2\right\} \quad . \qquad (60)
$$

Now we can define a new integration variable

$$
y \equiv \frac{x'}{d} + id\left(\frac{p'}{\hbar} - k\right) \quad . \qquad (61)
$$

At this point the integral has to be calculated in the complex plane along a line parallel to the real axis. By analiticity though, one can shift the path back to the real axis (in the variable y), so that

$$
\begin{aligned}
I &= \exp\left(-\frac{d^2}{2\hbar^2}(p'-\hbar k)^2\right) d\int_{-\infty}^{+\infty} dy \exp\left(-\frac{1}{2}y^2\right) \\
&= \sqrt{2\pi}d \exp\left(-\frac{d^2}{2\hbar^2}(p'-\hbar k)^2\right) \quad , \qquad (62)
\end{aligned}
$$

which completes our proof of formula (1.7.42).