FINAL EXAM (Due Jan 09, 2006)

Your final exam has to be delivered to the secretary of the Lorentz Institute (Oort building, 2nd floor, room 251) **before 2pm on jan 09**. For this final exam the full **Honor Code** is active. Your graded exam as well as your final grade are available at the secretaries office on Jan 24, 2006.

1. This excercise highlights the Landau level structure associated with electrons which are confined to move in a plane (the '2DEG's' realized in MOSFET's) under the influence of a large magnetic field.

   An electron moves in the x-y plane in the presence of a uniform magnetic field in the z-direction $\mathbf{B} = B\hat{z}$.

   a. Evaluate the commutator $[\Pi_x, \Pi_y]$ where $\Pi_x = p_x - \frac{eA_x}{c}$ and $\Pi_y = p_y - \frac{eA_y}{c}$.

   b. By comparing the Hamiltonian (neglecting the kinetic energy part in z-direction) and the commutation relation obtained in (a) with those of the one dimensional oscillator problem, show how we can immediately write the energy eigenvalues as,

   $$E_n = \hbar\omega_c(n + 1/2) \tag{1}$$

   where $\omega_c = |eB|/(mc)$ is the classical cyclotron frequency and n is a non negative integer including 0. The states with quantum number n are the Landau levels.

   c. Let us now consider the time evolution of the x and y coordinates of the particle. The starting point is Sakurai Eq. (2.6.22): $m dx(t)/dt = mv_x(t) = \Pi_x(t)$ and $m dy(t)/dt = mv_y(t) = \Pi_y(t)$, where $x(t)$ and $y(t)$ are the position operators for the x and y coordinates in the Heisenberg representation. In the standard harmonic oscillator problem the time evolution of the position and momentum operators is given by

   $$q(t) = \sqrt{\frac{\hbar}{2m\omega}}\left(ae^{-i\omega t} + a^\dagger e^{i\omega t}\right), \quad p(t) = -i\sqrt{\frac{m\omega\hbar}{2}}\left(ae^{-i\omega t} - a^\dagger e^{i\omega t}\right) \tag{2}$$

   where a and $a^\dagger$ are the annihilation/creation operators at $t = 0$. Use the analogy with the harmonic oscillator to demonstrate that

   $$\begin{aligned} \Pi_x(t) &= m\left(v_x(0)\cos(\omega_c t) + v_y(0)\sin(\omega_c t)\right), \\ \Pi_y(t) &= m\left(-v_x(0)\sin(\omega_c t) + v_y(0)\cos(\omega_c t)\right) \end{aligned} \tag{3}$$

   where $v_x(0), v_y(0)$ are the velocities at $t = 0$.

   d. It follows from the result of (c) that

   $$\begin{aligned} x(t) &= x_0 + \frac{v_x(0)}{\omega_c}\sin(\omega_c t) - \frac{v_y(0)}{\omega_c}\cos(\omega_c t) = x_0 - \frac{\Pi_y(t)}{m\omega_c} \\ y(t) &= y_0 + \frac{v_x(0)}{\omega_c}\cos(\omega_c t) + \frac{v_y(0)}{\omega_c}\sin(\omega_c t) = x_0 + \frac{\Pi_x(t)}{m\omega_c} \end{aligned} \tag{4}$$

   where x_0, y_0 is the position of the particle at $t = 0$. Use these relations to calculate $r^2(t) = [x(t) - x_0]^2 + [y(t) - y_0]^2$. What is the classical interpretation of $r^2(t)$ and (x_0, y_0)? Calculate also the expectation value of r^2 in the n-th Landau level. Does this have a classical interpretation?

e. The $x(t), y(t)$ and $p_x(t), p_y(t)$ should behave as canonical position and momentum operators satisfying $[x, y] = [p_x, p_y] = 0$ and $[x_i, p_j] = i\hbar\delta_{i,j}$. What does this imply for the 'guiding center coordinates' x_0, y_0? Consider the commutator $[x_0, y_0]$ and the commutators of x_0, y_0 with the Hamiltonian. Discuss what this means physically.

2. Even for the simple harmonic oscillator the propagator is quite cumbersome, and the answer is given by Sakurai Eq. (2.5.18):

$$\begin{aligned} K(x_1, t_1; x_0, t_0) &= \sqrt{\frac{m\omega}{2\pi i\hbar \sin\left(\omega(t_1 - t_0)\right)}} \\ &\quad \times \exp\left[\left(\frac{im\omega}{2\hbar\sin(\omega(t_1 - t_0))}\right) \times \left((x_1^2 + x_0^2)\cos(\omega(t_1 - t_0)) - 2x_1x_0\right)\right] \end{aligned}$$

Let us first investigate what this propagator tells about the time evolution of a coherent state wavefunction.

a. Given a coherent state $|\alpha\rangle$, demonstrate that one finds a normalized wavefunction in position representation

$$\psi_\alpha(x) = \langle x|\alpha\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \exp(-|\alpha|^2\cos^2\delta)\exp\left(-\frac{1}{2}\frac{m\omega}{\hbar}x^2 + \alpha\sqrt{\frac{2m\omega}{\hbar}}x\right), \tag{5}$$

where we have written $\alpha = |\alpha|e^{i\delta}$ and have dropped an unimportant phase factor. Hint: Start with the equation $\langle x|a|\alpha\rangle = \alpha\langle x|\alpha\rangle$, express a as a combination of $\hat{x}$ and $\hat{p}$, and solve the resulting differential equation for $\psi_\alpha(x)$.

b. Given that the wavefunction at time $t = 0$ is given by the $\psi_\alpha(x)$ obtained under (a), calculate the wavefunction at some later time t using the propagator of the harmonic oscillator. Show that

$$|\psi_\alpha(x,t)|^2 = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} \exp\left(-\frac{m\omega}{\hbar}\left(x - |\alpha|\sqrt{2}\sqrt{\frac{\hbar}{m\omega}}\cos(\omega t - \delta)\right)^2\right). \tag{6}$$

Discuss what this means physically.

Inspecting the expression for the propagator one infers that the dependence on x_0, x_1 is entirely in the exponential factor. Let us find out what this means using the path-integral. The Lagrangian for a harmonic oscillator with mass m and angular frequency ω in one spatial dimension is given by $L(\dot{x}, x) = \frac{m}{2}\left(\dot{x}^2 - \omega^2x^2\right)$.

c. An arbitrary path $x(t)$ can be written as the sum of the classical path $x_{cl}(t)$ and a quantum 'detour' $y(t)$: $x(t) = x_{cl}(t) + y(t)$. Show that the propagator can be written as,

$$\begin{aligned} K(x_1, t_1; x_0, t_0) &= \int_{x_0}^{x_1} \mathcal{D}x(t)e^{\frac{i}{\hbar}S[x(t)]} \\ &= e^{\frac{i}{\hbar}S_{Cl}[x_0(t_0), x_1(t_1)]} \times \int_{y=0}^{y=0} \mathcal{D}y(t)e^{\frac{i}{\hbar}S[y(t), t_1 - t_0]} \end{aligned} \tag{7}$$

where S_{cl} is the action associated with the classical path $x_{cl}(t)$. This implies that the dependence of the endpoints is entirely governed by the classical action while the contributions of non-classical paths add up to a factor which depends only on the time difference $t_1 - t_0$. Hints: (i) discuss why $\mathcal{D}x \to \mathcal{D}y$, (ii) use the stationary phase condition $\delta S = 0$ to show that the path integral factorizes.

d. Determine $x_{cl}(t)$ and $\dot{x}_{cl}(t)$ from the classical equation of motion derived from the Euler Lagrange equation:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0. \tag{8}$$

Compute the classical action using these results and demonstrate that S_{cl} is indeed responsible for the dependence of the propagator on the spatial coordinates.

3. In one dimensional physics quantum statistics is in a way non-existent and this freedom can be used to obtain quite counterintuitive insights. A most famous example is the Jordan-Wigner (JW) transformation: this maps the $S = 1/2$ spin chain on a problem of spinless fermions. The mapping itself is an interesting, and not too difficult exercise in second quantization.

The starting point is the one dimensional anisotropic Heisenberg spin chain' in terms of $S = 1/2$ spin operators S_i^α living on sites i, having nearest neighbor interactions J_α depending on the internal spin direction α,

$$\hat{\mathcal{H}} = -\sum_{i=-\infty}^{\infty}\left(J_x \hat{S}_i^x \hat{S}_{i+1}^x + J_y \hat{S}_i^y \hat{S}_{i+1}^y + J_z \hat{S}_i^z \hat{S}_{i+1}^z\right). \tag{9}$$

The basic idea of the JW transformation is to make use of the Pauli principle which tells us that we can only have zero or one spinless fermion at the same position. Since the Hilbert space of a single spin $\frac{1}{2}$ is two dimensional we can identify the $S_z = -\frac{1}{2}$ state with the no-fermion state and the $S_z = \frac{1}{2}$ state with the one-fermion state:

$$\begin{aligned} |\downarrow\rangle &= |0\rangle = c|1\rangle, \\ |\uparrow\rangle &= |1\rangle = c^\dagger|0\rangle. \end{aligned} \tag{10}$$

Obviously this identification corresponds to (in this exercise we use $\hbar = 1$)

$$\hat{S}^z = c^\dagger c - \frac{1}{2}, \quad \hat{S}^- = c, \quad \text{and } \hat{S}^+ = c^\dagger. \tag{11}$$

a. Show that these relations between the spin operators and the Fermi operators are consistent with the spin commutator relations $[\hat{S}^\alpha, \hat{S}^\beta] = i\epsilon_{\alpha\beta\gamma}\hat{S}^\gamma$ $(\alpha, \beta, \gamma \in \{x, y, z\})$.

b. Unfortunately, fermions on different sites anticommute whereas the spins should commute. Thus, one should modify the mapping (11) to transform a non-local anticommutation into a commutation without changing the local commutation relations. In one dimension the solution was found by Jordan and Wigner. One attaches a string of operators to each fermion. The string produces the needed minus sign. The mapping becomes

$$\hat{S}_i^+ = c_i^\dagger \exp(i\pi \sum_{j=-\infty}^{i-1} c_j^\dagger c_j), \tag{12}$$

$$\hat{S}_i^- = \exp(-i\pi \sum_{j=-\infty}^{i-1} c_j^\dagger c_j) c_i, \tag{13}$$

$$\hat{S}_i^z = c_i^\dagger c_i - \frac{1}{2}. \tag{14}$$

Show that these operators fulfill the spin commutator relations $[\hat{S}_i^\alpha, \hat{S}_j^\beta] = i\delta_{ij}\epsilon_{\alpha\beta\gamma}\hat{S}_i^\gamma$. Hint: show first that $\exp(\pm i\pi \sum_{j=-\infty}^{i-1} c_j^\dagger c_j) = \Pi_{j<i}(1 - 2c_j^\dagger c_j)$.

c. Use the JW transformation (14) to show that the spin Hamiltonian (9) can be written in terms of spinless fermions as

$$\begin{aligned}\hat{\mathcal{H}} &= -\sum_i \Big[t(c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1}) + \Delta(c_{i+1}^\dagger c_i^\dagger + c_i c_{i+1}) \\ &\qquad + V(c_i^\dagger c_i - \frac{1}{2})(c_{i+1}^\dagger c_{i+1} - \frac{1}{2}) \Big], \end{aligned} \tag{15}$$

where $t = (J_x + J_y)/4$, $\Delta = (J_y - J_x)/4$, and $V = J_z$.

d. We have mapped the spin Hamiltonian (9) to a chain of spinless interacting fermions. Even in the presence of V much is known, but this is well beyond the scope of this course. Therefore, from now on we focus on the case of an anisotropic XY-spin chain ($J_z = 0$). In this special case life is much easier since (15) reduces to a Hamiltonian of free fermions which can be diagonalized using the tools we have learned in this course. Transform to momentum basis $c_j^\dagger = \frac{1}{N}\sum_q d_q^\dagger e^{iqx_j}$ (we assume N sites subject to periodic boundary conditions), consider the fermions with momenta $q > 0$ and $q < 0$ separately, and show that the Hamiltonian can be written as

$$\hat{\mathcal{H}} = \sum_{q>0} \Big[\epsilon_q (d_q^\dagger d_q - d_{-q} d_{-q}^\dagger) + i\Delta_q (d_q^\dagger d_{-q}^\dagger - d_{-q} d_q) \Big], \tag{16}$$

where $\epsilon_q = -2t\cos(qa)$ and $\Delta_q = -2\Delta\sin(qa)$. Here a denotes the lattice spacing.

e. Find the fermionic Bogoliubov transformation

$$\begin{pmatrix} a_q \\ a_{-q}^\dagger \end{pmatrix} = \begin{pmatrix} u & v \\ -v^* & u^* \end{pmatrix} \begin{pmatrix} d_q \\ d_{-q}^\dagger \end{pmatrix} \tag{17}$$

diagonalizing the Hamiltonian and show that the dispersion associated with the excitations of the spin chain is given by

$$\omega_q = \frac{1}{2}\sqrt{J_x^2 + J_y^2 + 2J_xJ_y\cos(2qa)}. \tag{18}$$

Sketch this result and discuss briefly the cases $J_x = J_y \neq 0$ and $J_x = 0 \neq J_y$.