Quantum Theory, Fall 2005

Assignment 6. (Due Dec. 7, 2005)

1. In the second quantization of the free electromagnetic field, show explicitly that the field Hamiltonian $\mathcal{H}_\gamma = \sum_{\mathbf{k},\lambda} \hbar\omega_k (a^\dagger_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda} + 1/2)$ can also be obtained by computing the expression:

$$\mathcal{H}_\gamma = \frac{1}{8\pi} \int d^3\mathbf{r}\, (\mathbf{E}^2 + \mathbf{B}^2)\,,$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic field Heisenberg operators introduced in Gottfried, Section 52.1, Eq.'s (13, 14).

Hints: You may use that $\frac{1}{V}\int d^3\mathbf{r}\, \exp\left(i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}\right) = \delta_{\mathbf{k},\mathbf{k}'}$. You may also need the following relation: $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$.

2. Consider a chain of atoms with masses m, connected by springs with spring-constant γ, and Hamiltonian

$$H_{ph} = \sum_{n=-\infty}^{\infty} \left(\frac{p_n^2}{2m} + \frac{\gamma}{2}(u_{n+1} - u_n)^2 \right), \tag{1}$$

where u_n are the displacements of the atoms from their equilibrium positions, and p_n are the corresponding conjugate momenta. Consider the problem in quantum-mechanics, i.e. treat u_n and p_n as operators satisfying the canonical commutation relation $[u_n, p_{n'}] = +i\hbar\delta_{n,n'}$.

   a. Diagonalize the Hamiltonian (1). First perform a Fourier transformation $u_n \to u_k$ and $p_n \to p_k$, and determine the dispersion ω_k of the lattice vibrations. Introduce subsequently creation and annihilation operators by $u_k \sim a_k + a^\dagger_{-k}$ and $p_k \sim -i(a_k - a^\dagger_{-k})$. Write the Hamiltonian (1) in terms of the $a^\dagger_k$'s and a_k's, creating quanta of lattice vibration: the 'phonons'. What is the ground state energy of the system?

   b. The 'springs' between atoms will in reality be anharmonic, and we can add this as a perturbation to the Hamiltonian,

$$H'_{ph} = \sum_{n=-\infty}^{\infty} \gamma'(u_{n+1} - u_n)^3, \tag{2}$$

   assuming that γ' is small. Rewrite the Hamiltonian (2) in terms of the phonon operators $a^\dagger_k, a_k$. The phonons can be viewed as particles (like photons). Explain why anharmonicity turns into interaction between the phonons, and discuss the nature of this interaction. What can you say about the momentum conservation of the phonons in the presence of the anharmonicity (2).

3. Calculate the lifetime of a hydrogen atom in its $2p$ state. Since you are considering a spontaneous emission process, you must work with the quantized electromagnetic field.

The initial state is $|i\rangle = |nlm\rangle \otimes |0\rangle$ where $|nlm\rangle$ is the $2p$ state of the hydrogen atom ($n = 2$, $l = 1$) and $|0\rangle$ the vacuum of the electromagnetic field. The final state is $|f\rangle = |100\rangle \otimes |\mathbf{k}\boldsymbol{\lambda}\rangle$ where the hydrogen atom is in the ground state and a $\mathbf{k}\boldsymbol{\lambda}$ photon was created.

a) Express the matrix element $\langle f|\, \mathbf{A}(\mathbf{r}) \cdot \mathbf{p}\, |i\rangle$ in terms of $\langle 100|\, \mathbf{r}\, |nlm\rangle$.
Hints: Use the dipole approximation for long wavelengths, $\mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}} \approx 1$. You may also need that $[\mathbf{r}, H_0] = \dfrac{i\hbar\mathbf{p}}{m}$ where H_0 is the Hamiltonian of the hydrogen atom (prove this identity).

b) Write down the transition rate $W_{i\longrightarrow f}$ using Fermi's golden rule. For the moment, keep the expression in terms of $\delta(\hbar\omega_k - \Delta E)$, where $\Delta E = E_{nlm} - E_{100}$ and $\hbar\omega_k$ is the energy of the emitted photon.

c) Expand the product $\mathbf{r} \cdot \boldsymbol{\lambda}$ in spherical harmonics Y_{lm}. Use that

$$x = \frac{1}{2}\sqrt{\frac{8\pi}{3}}\, r\,(-Y_{11} + Y_{1-1}), \qquad y = \frac{i}{2}\sqrt{\frac{8\pi}{3}}\, r\,(Y_{11} + Y_{1-1}), \qquad z = \sqrt{\frac{4\pi}{3}}\, r\, Y_{10}.$$

Rewrite $\mathbf{r} \cdot \boldsymbol{\lambda}$ as $\mathbf{r} \cdot \boldsymbol{\lambda} = \sqrt{\dfrac{4\pi}{3}}\, r\,(-\lambda_{-1}Y_{11} - \lambda_1 Y_{1-1} + \lambda_0 Y_{10})$.

d) Show that $\langle 100|\, \boldsymbol{\lambda}\cdot\mathbf{r}\, |nlm\rangle = \dfrac{1}{\sqrt{3}} \displaystyle\int_0^\infty r^3 R_{10}(r) R_{21}(r)\, \mathrm{d}r\, (\lambda_1\delta_{m,1} + \lambda_0\delta_{m,0} + \lambda_{-1}\delta_{m,-1})$,
where $R_{nl}(r)$ is the radial component of the hydrogen wave function.
Hints: You may use that $Y_{00} = \dfrac{1}{\sqrt{4\pi}}$ and $Y_{l,-m} = (-1)^m (Y_{l,m})^*$.

e) Average the transition rate over all three initial m states and use that
$\displaystyle\int_0^\infty r^3 R_{10}(r) R_{21}(r)\mathrm{d}r = \sqrt{\frac{3}{2}\frac{2^8}{3^5}}\,\frac{\hbar^2}{m\,e^2}$.

f) Sum over all possible photon momenta and the two polarizations to obtain:

$$W_{i\longrightarrow f} = \left(\frac{2}{3}\right)^8 \alpha^5\, \frac{m\,c^2}{\hbar}\,,$$

where $\alpha = \dfrac{e^2}{\hbar c}$. Using this result, compute the lifetime of the hydrogen atom in the $2p$ state in seconds.