Assignment 3. (Due Oct 26, 2005)

1. Precessing spins in the Heisenberg picture (read ch. 2.2 carefully !):

   a. Start with Sakurai 2.1 and find the Heisenberg operators for $S_a(t)$ with $a = x, y, z$ as stated in the problem.

   b. Assume that a system is in either the $|x+\rangle$ or $|x-\rangle$ Heisenberg states. Calculate the expectation values of $S_a(t)$ with $a = x, y, z$ for these states and show that they agree with Eqs. (2.1.61, 2.1.62).

   c. Find the Heisenberg basis kets $|x\pm, t\rangle$ corresponding to the basis kets $|x\pm\rangle$ at $t = 0$.

   d. Assume that a system is in either the $|x+\rangle$ or $|x-\rangle$ Heisenberg state. Calculate the probability $|\langle x\pm, t|x\pm\rangle|^2$ that a measurement of $S_x(t)$ at $t > 0$ will result in an eigenvalue equal to the one at $t = 0$.

   e. Like (d) but evaluate now $|\langle x\mp, t|x\pm\rangle|^2$, i.e. the measurement yields an eigenvalue different from the one at $t = 0$.

   f. Calculate the base kets $|y\pm, t\rangle$ of $S_y(t)$ and the probabilities $|\langle y\pm, t|x\pm\rangle|^2$ and $|\langle y\mp, t|x\pm\rangle|^2$ that a Heisenberg state $|x\pm\rangle$ will have eigenvalues $\pm\hbar/2$ of $S_y(t)$ at $t > 0$.

2. Eq. (2.2.30) implies that a wave packet spreads in time. We want to understand this by studying the behavior of a Gaussian wave packet, Eq. (1.7.35). We start by setting $k = 0$ and take $\hbar = 1$ to simplify the manipulations; take $V(x) = 0$.

   a. What is the time dependent wave function $\phi_\alpha(p, t) = \langle p|\alpha, t\rangle$ in the p-representation corresponding to Eq. (1.7.42)?

   b. Use this result to find the time dependent dispersion of x, $\langle(\Delta x)^2\rangle_t$. Note: if you don't know how to represent the x operator in the p representation do exercise Sakurai 1-33(a).

   c. Use (a) to find the time dependent wave function $\psi_\alpha(x, t) = \langle x|\alpha, t\rangle$ (x-representation), corresponding to Eq. (1.7.35).

   d. Use (c) to find the time dependent dispersion $\langle(\Delta x)^2\rangle_t$. Does this agree with (b)?

   e. How does (d) compare with eq. (2.2.30)?

   f. Can you explain *qualitatively* why the dispersion in x increases?

   g. Explain *qualitatively* what would change when we would have considered the case that $k \neq 0$.

   h. What is the time dependent dispersion of p, $\langle(\Delta p)^2\rangle_t$?

   j. What is the expression corresponding to Eq. (2.2.30) for the time-dependence of the dispersion in p? Does the result for $\langle(\Delta p)^2\rangle_t$ agree with this?

3. Consider the harmonic oscillator Hamiltonian Eq. (2.3.1.). At time $t = 0$ the state $|\alpha\rangle = (|n\rangle + |n+1\rangle)/\sqrt{2}$ is prepared. Evaluate the expectation values $\langle\alpha|H(t)|\alpha\rangle$, $\langle\alpha|x(t)|\alpha\rangle$, and $\langle\alpha|p(t)|\alpha\rangle$ at later times t.

4. Consider a one-dimensional simple harmonic oscillator with angular frequency ω_0. For $t < 0$, the system is in the ground state. For $t > 0$, there is also a time-dependent potential:

$$V(t) = F_0\, x\, \cos\omega t$$

where F_0 is constant in both space and time. Obtain an expression for the the expectation value $\langle x \rangle$ as a function of time using time-dependent perturbation theory to lowest non-vanishing order. Is this procedure valid for $\omega \approx \omega_0$?

Hint: You may use $\langle n' \,|x|\, n\rangle = \sqrt{\dfrac{\hbar}{2m\omega_0}}\left(\sqrt{n+1}\,\delta_{n',n+1} + \sqrt{n}\,\delta_{n',n-1}\right)$.

5. Consider a composite system made up of two spin 1/2 particles. For $t < 0$, the Hamiltonian does not depend on spin and can be taken to be zero by suitably adjusting the energy scale. For $t > 0$, the Hamiltonian is given by

$$H = \left(\frac{4\Delta}{\hbar^2}\right) \mathbf{S}_1 \cdot \mathbf{S}_2$$

where $\mathbf{S}_1$ and $\mathbf{S}_2$ are the spin operators of particle 1 and 2, respectively.

Suppose that the system is in the state $|{+}{-}\rangle$ for $t \leq 0$. Find, as a function of time, the probability for the system to be found in each of the following states $|{+}{+}\rangle$, $|{+}{-}\rangle$, $|{-}{+}\rangle$ and $|{-}{-}\rangle$.

a) By solving the problem exactly.
Hints: Express the Hamiltonian in terms of the total spin operator $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$. Find its eigenvalues and eigenvectors. Express the states $|{+}{-}\rangle$ etc. in terms of $|sm\rangle$.

b) By solving the problem assuming the validity of time-dependent perturbation theory with H as a perturbation switched on at $t = 0$. Derive the probabilities to leading (non-zero) order. Under what condition does (b) give the correct results?

Notation: The state $|+,-\rangle$ is an eigenstate of $\mathbf{S}_1^2$, $\mathbf{S}_2^2$, S_{1z} and S_{2z}:

$$S_{1z}\,|+,-\rangle = +\hbar/2\,|+,-\rangle\,, \qquad S_{2z}\,|+,-\rangle = -\hbar/2\,|+,-\rangle\,.$$

The state $|sm\rangle$ is an eigenstate of $\mathbf{S}_1^2$, $\mathbf{S}_2^2$, $\mathbf{S}^2$ and S_z:

$$\mathbf{S}^2\,|sm\rangle = \hbar^2\, s(s+1)\,|sm\rangle\,, \qquad S_z\,|sm\rangle = \hbar\, m\,|sm\rangle\,.$$