Quantum Theory, Fall 2005

Assignment 2. (Due Oct 12, 2005)

1. A spin 1/2 system is known to be in an eigenstate of $\mathbf{S} \cdot \hat{\mathbf{n}}$ with eigenvalue $\hbar/2$ where $\hat{\mathbf{n}}$ is a unit vector lying in the xz-plane that makes an angle γ with the positive z-axis.

   a. Suppose S_x is measured. What is the probability of getting $+\hbar/2$?

   b. Evaluate the dispersion in S_x, that is $\langle (S_x - \langle S_x \rangle)^2 \rangle$. For your own peace of mind check your answers for the special cases $\gamma = 0, \pi/2$ and π.

2. Find the linear combination of $|+\rangle$ and $|-\rangle$ kets that maximize the uncertainty product

$$\langle (\Delta S_x)^2 \rangle \langle (\Delta S_y)^2 \rangle. \tag{1}$$

   Verify explicitly that for the linear combination you found, the uncertainty relation for S_x and S_y is not violated.

3. When we first learn about quantum mechanics we sometimes hear that "if the momentum is known with precision (zero uncertainty) then the Heisenberg uncertainty principle requires then there to be infinite uncertainty in the position". For the x and p operators in the interval ($-\infty < x < \infty$) there is no problem with this, since the wavefunction with definite momentum $p = \hbar k$ is given by $\psi(k) \sim \exp(ikx)$, and the uncertainty in position for this wave function is definitely infinite. Now think about the HUP (Sakurai, Eq. 1.4.53) for a case in which the spectrum (set of eigenvalues) of the observable is bounded, e.g., the spin 1/2 system. Then it is impossible for an operator, e.g., S_x to have an infinite uncertainty. However, $\langle (\Delta A)^2 \rangle$ can be zero because the system is in an eigenstate of A , where A is e.g. S_z. Naively we might fear that Eq. (1.4.53) is violated since the commutator $[S_x, S_z] \neq 0$.

   a. What actually happens when you evaluate both sides of Eq. (1.4.53) for a system that is in an eigenstate $|+\rangle$ or $|-\rangle$ of $A = S_z$ with $B = S_x$. Is Eq. (1.4.53) correct?

   b. In the proof of Eq. (1.4.53), an anticommutator term was dropped from the RHS (see Eq. 1.4.63), because its inclusion would have only made the inequality stronger. Does this anticommutator causes any trouble with the case in which the system is in an eigenstate of $A = S_z$ and you have $B = S_x$? Why?

4. Assume that the Dirac δ function has the properties $\delta(x) = 0$ for all $x \neq 0$, $\int \delta(x) dx = 1$ and $\int f(x)\, \delta(x)\, dx = f(0)$. If a is a constant, what are:

   a. $\int f(x)\, \delta(ax)\, dx$?

   b. $\int f(x)\, \delta(x^2 - a^2)\, dx$?

   c. $\int f(x)\, \delta(g(x))\, dx$, where $g(x_0) = 0$ and $g(x) \neq 0$ for $x \neq x_0$?

   d. $\int f(x)\, (d\delta(x)/dx)\, dx$ (Yes – this is a derivative of a δ function !)

5. The fact that x and p satisfy the canonical commutation relation $[x, p] = i\hbar$ implies that the spectra of both the position and the momentum operators include all real numbers, i.e., $(-\infty < x < \infty)$ and $(-\infty < p < \infty)$. You can show this with the following considerations:

   a. Use the commutation relation for x and p to show that $[x, F(p)] = i\hbar\partial F/\partial p$ where F is an operator valued function of its argument.
   Hint: expand $F(p)$ in a power series and do not use the fact that the x operator is a derivative in the p representation.

   b. Now consider the operator $F(p) = e^{-ipa/\hbar}$, where a is any real number. Show that $F(p)|x'\rangle$ is an eigenvector of x with eigenvalue $x'+a$. Since a can be any real number, any real number can be an eigenvalue of the operator x, QED!

   c. In order to show that the spectrum of p is any real number, find $[p, G(x)]$ and follow part b.

   d. How does the particle confined between two rigid walls (Sakurai 1-21) manage to avoid having eigenstates for all real values of p?

6. Consider the Gaussian wave packet, Sak. Eq. (1.7.35).

   a. Use this expression to calculate the dispersion of p for this state, i.e., verify Eq. (1.7.40).

   b. Check your answer using Eq. (1.7.42).

   c. Show that the second line in Eq. (1.7.42) follows from the first.

Notabene: you should remember and be able to derive the following useful "Gaussian" integrals:

i $\int_{-\infty}^{\infty} dx e^{-x^2} = \sqrt{\pi}$

ii $\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} dx e^{-\frac{(x-\mu)^2}{2\sigma^2}} = 1$

iii $\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} dx x e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \mu$

iv $\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} dx (x-\mu)^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \sigma^2$