Assignment 1. (Due Sept. 28, 2005)

1. A particle in a one dimensional box, and Dirac brackets.

a. Consider the problem of a quantum mechanical particle confined in an one dimensional box. Consider a Schrödinger equation,

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) \tag{1}$$

with a potential $V(x) = 0$ in the interval $-a \leq x \leq a$, and infinite elsewhere. Derive the complete spectrum of energy eigenstates $|n\rangle$ with wavefunctions in position representation $\Psi_n(x)$ and eigenvalues E_n .

b. Using 24-th century quantum technology an experimentalist manages to prepare a state $|\Phi\rangle$ in this box corresponding in x representation with a wavefunction $\Phi(x) = N(a - |x|)$, where N is a normalization constant which is yet to be determined. Evaluate the overlap matrix elements $\langle n|\Phi\rangle$ of this state with the energy eigenstates determined in question (1.a).

c. Calculate the overlap $\langle x|\Phi\rangle$ where $|x\rangle$ refers to the eigenstates of the position operator x of the particle in the box.

d. Calculate the overlap matrix elements $\langle x|x'\rangle$ where $|x\rangle, |x'\rangle$ both refer to position eigenstates.

2. Using the rules of bra-ket algebra, prove or evaluate the following:

a) $(XY)^\dagger = Y^\dagger X^\dagger$, where X and Y are operators.

b) $\mathrm{tr}(XY) = \mathrm{tr}(YX)$,
where $\mathrm{tr}(XY) = \sum_{a'} \langle a'| XY |a'\rangle$ with $\{|a'\rangle\}$ a complete basis set, and X and Y are operators.

c) $\sum_{a'} \Psi^*_{a'}(\mathbf{x}')\Psi_{a'}(\mathbf{x}'')$, where $\Psi_{a'}(\mathbf{x}') = \langle \mathbf{x}'|\, a'\rangle$

3. Using the orthonormality of $|+\rangle$ and $|-\rangle$, prove

$$[S_i, S_j] = i\varepsilon_{ijk}\hbar S_k,$$
$$\{S_i, S_j\} = \frac{\hbar^2}{2}\delta_{ij},$$

where

$$S_x = \frac{\hbar}{2}\left(|+\rangle\langle-| + |-\rangle\langle+|\right),$$
$$S_y = \frac{i\hbar}{2}\left(-|+\rangle\langle-| + |-\rangle\langle+|\right),$$
$$S_x = \frac{\hbar}{2}\left(|+\rangle\langle+| - |-\rangle\langle-|\right).$$

4. Construct $|\mathbf{S}\cdot\hat{\mathbf{n}};+\rangle$ such that

$$\mathbf{S}\cdot\hat{\mathbf{n}}|\mathbf{S}\cdot\hat{\mathbf{n}};+\rangle = \frac{\boldsymbol{\hbar}}{\mathbf{2}}|\mathbf{S}\cdot\hat{\mathbf{n}};+\rangle \tag{2}$$

in terms of the S_z up- and down $S = 1/2$ states $|\pm\rangle$, while $\hat{\mathbf{n}}$ is parametrized in terms of the Euler angles θ, ϕ as $\hat{\mathbf{n}} = (sin\theta cos\phi, sin\theta \sin\phi, cos\theta)$. N.B.: this is an elementary example of a generalized coherent state for $S = 1/2$.

5. A certain observable in quantum mechanics has a 3×3 matrix representation as follows:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{3}$$

a. Find the normalized eigenvectors of this observable and the corresponding eigenvalues. Is there any degeneracy?

b. Give a physical example where all of this is relevant.

6. Consider the following Hamiltonian for a 2-state system:

$$H = a(|1\rangle\langle1| - |2\rangle\langle2| + |1\rangle\langle2| + |2\rangle\langle1|),$$

where a is a number with the dimension of energy. Find the energy eigenvalues and the corresponding energy eigenkets (as linear combinations of $|1\rangle$ and $|2\rangle$). $|1\rangle$ and $|2\rangle$ form a complete orthonormal basis set.