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2.4 A modern, but way too short history of the theory of superconductivity at a high temperature.

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1.1 Introduction

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One cannot write the history of a war when it is still raging and this is certainly applying to the task we are facing dealing with the theory of superconductivity in the post BCS era. The BCS theory was of course a monumental achievement that deserves to be counted among the greatest triumphs in physics of the twentieth century. With the discovery of high- T_c superconductivity in the cuprates in 1986¹ a consensus emerged immediately that something else was at work than the classic (i.e. phonon driven) BCS mechanism. A quarter of a century later this subject is still contentious, perhaps even more so than at any instance in the past. It is plainly impossible to write an 'objective' history of any kind and this piece has no pretense in this regard. I thought it would be useful to write instead some sort of eyewitness account, serving two potential readerships: in the first place the newcomers who wonder about the prehistory of some standard notions that appear as established and controversial at the same time. The other customers are right now no more than a potentiality. Imagine that the theory of high- T_c superconductivity will turn truly glorious at some point in the future. The era that I am discussing here will then have a similar status as the 'dark ages' that preceded the discovery of BCS theory as described elsewhere in this volume. I can imagine that an eyewitness account like the present one might be entertaining for the professional historians who want to chronicle this triumph of 21-th century physics.

I am aware that this exposition is far from complete – there are quite a number of ideas that are very interesting by themselves, while they might be on the long run more consequential than the matters I discuss here. However, given the length restrictions I just have not the space to give a truly

¹K.A. Müller, G. Bednorz, Z. Phys. B64 (1986) 189.

comprehensive overview of everything that has been explored. I have to restrict myself to a sketch of the history behind the notions that are right now most on the foreground. Similarly, I do not claim rigor with regard to distributing the credits to the various individuals involved. Although at some instances I will draw attention to contributions that I find are underexposed, this material is surely not intended as background material for award committees.

This eyewitness account is also colored with regard to the substances that are discussed. Whether we like it or not, humanity is a religious species and belief systems play a big role when science is still in the making. One better be aware of it and in physics I like to be myself a 'denomination tourist'. I have dwelled around among the various belief systems, getting into it to a degree that I felt myself to be a believer, to hop at that point to a next system. In the course of time I have settled into a faith that is right now quite popular: the 'agnostic denomination in high- T_c superconductivity'. Their dogma is as follows: "the research in unconventional superconductivity is flourishing. It is however owned by the experimentalists. Energized by the high- T_c puzzle, the experimentation in quantum matter leaped forward in the last twenty years, due to a variety of highly innovative instrumental developments. These new experimental facts have however just deepened the mystery for the theorists. The agnostic zeal is anchored in the conviction that these fantastic new data are begging for an explanation in terms of mathematical equations of a beauty and elegance that will measure up to the products of Einstein and Dirac. Right now we have not the faintest idea what these equations are but that should not keep us away from attempting to find them."

Surely this has imprinted on the organization of this text. I will discuss matters in a sequence that is in a historical order (from old to new), while at the same time the sense of the awareness of dealing with a mystery (the agnostic dogma) is increasing.

The point of departure is the highly influential 'denomination' that rests on the believe that the theoretical framework as emerged in the 1950's in essence still suffices to explain the physics of the cuprates and so forth. The normal state is a Fermi-liquid that is perhaps a bit obscured by perturbative processes. However, the essence of BCS theory is still fully intact and to explain the high- T_c superconductivity one needs a 'superglue' that has more punch than the traditional phonons. This superglue is than supposed to be embodied by spin fluctuations. Since this 'fifties paradigm' has been around for a long time it has turned into a framework that pretends to be quite qualitative. This "holy trinity" (Local Density Approximation (LDA) band structure, Random Phase Approximation (RPA) spin fluctuation superglue, Eliashberg equations for the superconductivity) is right now in the middle of a revival in the context of the superconductivity in the iron pnictides. It surely should be taken quite seriously but more than anywhere else the danger of religious illusions is around the corner. In this regard I perceive it as beneficial to be aware of the long, and contentious history of the spin fluctuation superglue idea. This is the narrative of section 2.4.2.

Although it has been a bit on the retreat recently, the next influential school can be called 'RVB, and related phenomena'. This refers to a highly imaginative set of ideas, that emerged in the wild early years of high- T_c superconductivity under the inspired intellectual leadership of Phil Anderson. Regardless whether it has anything to say about the real life of high- T_c superconductors these deserve attention if not only because it developed into a quite interesting and innovative branch of theoretical physics.

The spin fluctuation theories are rooted in the perturbative schemes of the fermiology that was established in the 1950's. The Resonating Valence Bond (RVB) ideas eventually rest on the recognition that doped Mott insulators have to be different. This story starts with the last skirmishes of a war that is much older: the understanding of the Mott-insulating state itself. As I will discuss in section 2.4.3, this pursuit started a long time ago, while it settled in the late 1980's when, as a byproduct of the high- T_c research, a rather detailed and complete understanding of the real life Mott insulators was established. Section 2.4.4 is dealing with the RVB school of thought. This started from an appealing idea by Phil Anderson that evolved during the wild early years of high- T_c in something of a mathematical science by itself. Different from the 1950's spin fluctuations, these 'gauge theories of strongly correlated fermions' rest conceptually on the discoveries in the 1970's

in high energy physics. Although it still lacks both convincing experimental support and real mathematical control, these theories surely serve well the purpose of confusing the high energy theorists. At stake is the meaning of the gauge principle. In fundamental physics it is a primordial principle. The 'RVB' gauge theories are insisting however that the gauge principle is governed by collectivity, telling how large numbers of 'un-gauged' microscopic degrees of freedom co-operate in unison to generate 'gauged' collective behaviors.

Keeping the historical order, in section 2.4.5 I will focus on the subject that has been dominating the field during the last ten years or so: the physics of the underdoped 'pseudogap regime'. This has been so much on the foreground because it turned out that there was much to discover using the modern experimental techniques. It is now clear that there is a collection of exotic competing orders at work, including stripes, quantum liquid crystals and spontaneous diamagnetic currents. Apparently theoretical physics as we know it does give a grip on orderly things, because theorists did make a big difference in this particular pursuit.

In the last section I will discuss what may be the beginning of the history of the future of high- T_c superconductivity. This revolves around the idea that the quantum physics of the mysterious superconductors is in the grip of the overarching symmetry principle of scale invariance. In analogy with the classical critical state, this 'quantum criticality' is believed to be caused by a phase transition happening at zero temperature, driven by quantum fluctuations. Given the increased understanding of the competing orders at work in the pseudogap phase this notion is at present becoming quite popular in the cuprates. In the context of the heavy fermions it has been however obvious for a long time that their superconductivity is closely related to quantum phase transitions and at stake is whether these share the essential physics with the cuprates. This is the stronghold for the agnostic theorists because nowhere else it is so obvious that we are lacking mathematical machinery that works as in the context of quantum critical 'fermionic' metals. I will end this chronicle on an optimistic tone by expressing my hope that the string theorists might actually have such machinery in the offering.

1.2 Pushing BCS to the limits: the spin fluctuation superglue.

The BCS theory is a remarkable achievement. A constant factor in the debate during the last twenty five years has been whether it is at all possible to get around the essence of BCS to explain superconductivity at a high temperature. Its central wheel is the Cooper instability which in turn rests on the physics of nearly free fermions. A first condition is that the normal state closely approaches the Fermi-liquid fixed point near the superconducting transition temperature. A second condition is that the net residual quasiparticle interactions turn attractive in the appropriate pairing channel. At this instance the Fermi-surface exerts its power, by singling out the pairing channel as uniquely singular. Feeding a generic fermion system with a very strong attractive interaction that overwhelms the Fermi-energy would just cause a clumping of the particles since in (nearly) classical systems there is no special stability associated with two particle bound states². The remainder of BCS is simple. Cooper pairs are bosons and the Cooper pairs Bose condense at the moment they form, of course modulo the possibility of 'dangerous' thermal order parameter fluctuations.

The question is whether the normal state is sufficiently like a Fermi-liquid for the BCS mechanism to apply directly. This has been dividing the minds more than anything else – the 'radical' theories discussed in the next sections depart from the assertion that the normal state is something else than a Fermi liquid. Assuming the governance of the Fermi-liquid, the only issue is then to explain the

²Although it is usually worked under the rug, this problem hunts ideas involving local pairing, like the bipolarons.

origin of the attractive interactions. This is far from trivial since electrons actually repel each other rather strongly on the microscopic scale through the Coulomb interactions. Much of the hard work of diminishing these bare interactions can be ascribed to the magic of the Fermi-liquid. The ubiquitous nature of the quasiparticle gas is much more of a mystery than is often realized. The particular limiting case where on the microscopic scale the interactions are substantially smaller than the bare Fermi energy is nowadays well understood, although this involves the fanciful 'Polchinski-Shankar' functional renormalization group. By principle one has to keep track of an infinity of coupling constants 'spanning up' the Fermi-surface, when integrating out short distance degrees of freedom. In the renormalization process, one finds that besides the 'harmless' marginality of the Landau F functions, one has only to watch a possible (marginal) relevancy in the Cooper channel.³ The bottom line is that with the exception of the BCS superconductivity the Fermi-liquid is generically (i.e., away from special nesting conditions) stable as an infrared fixed point.

The problem is however that the systems of interest in the present era are invariably of the kind where the microscopic interaction scale exceeds the bare kinetic energy by order(s) of magnitude. One can no longer rely on the perturbative renormalization group and it is just an experimental fact that Fermi-liquids are nevertheless ubiquitous. A classic example is ³He. At microscopic distances this is like a quantum version of a dense, highly correlated Van der Waals liquid. However, upon cooling it down to milli-kelvins one finds a long wavelength physics corresponding with an impeccable Fermi-gas of quasiparticles. The miracle is that the 'hard Helium balls' have become effectively completely transparent relative to each other, only communicating via the Pauli principle, while the effects of interactions is that these 'quasi Helium atoms' are three times heavier than the real atoms. I learned from Robert Schrieffer that in the 1960's-70's theorists tried hard to explain this wonder, not getting anywhere.⁴ Fermi-liquids are unreasonably resilient and this is perhaps the best reason to take the 'conservative' theories of high- T_c superconductivity serious.

The next question becomes: how to turn the residual weak quasiparticle repulsions of the Fermi-liquid into serious attractive interactions? The classic BCS answer is of course phonons. The exchange of bosons that are not governed by the gauge principle will generically give rise to an induced attractive interaction. The phonons are just perfect: they are external to the electron system since they originate in the lattice, their interactions with the quasiparticles emerge naturally and they have the added benefit of causing retarded interactions. Phonon energies are tiny compared to the Fermi energy of conventional metals and the associated 'Migdal parameter' can be exploited to simplify the perturbation theory into a virtually exact 'resummed' Migdal-Eliashberg theory.

This control over the classic phonon-driven BCS theory explains the sociological earthquake that rumbled through the physics community following the Bednorz-Müller discovery of superconductivity at 34K in $La_{2-x}Ba_xCuO_4$ in 1986. Back then it was believed that phonons could not be responsible for a T_c exceeding 30 K and a consensus emerged immediately that something new was at work. In hindsight this is actually a bit embarrassing. It is a number game and numbers are dangerous. The strongest claim rested on stability considerations⁵ that were subsequently challenged by theorists of Ginzburg's high- T_c superconductivity group at the Lebedev institute⁶. In fact, in 2001 superconductivity at 40 K was discovered in the simple MgB_2 system. In no time a consensus emerged that the superconductivity is phonon driven, with a number of material details (high phonon frequencies, multi-bands) conspiring to optimize the phonon mechanism, while the phonons responsible for the superconductivity are unrelated to the factors rendering the crystal to be

³For a fanciful recent discussion see S.A. Parameswaran, R. Shankar and S.L. Sondhi, arXiv:1008.2492.

⁴Remarkably, as a very recent development it appears that the quantum-fields gravity holographic dualities of string theory shed a new light on this old problem: the Fermi liquid is related to the gravitational stability of special black holes with 'fermion hair', see section 2.4.5.

⁵M.L. Cohen and P.W. Anderson, in *Superconductivity in d- and f-band Metals*, ed. D.H. Douglass, AIP, New York (1972), p. 17.

⁶O.V. Dolgov, D.A. Kirzhnits and E.G. Maksimov, Rev. Mod. Phys. 53 (1981) 81.

stable. This development gave also credibility to the claim that such phonon driven superconductivity can be quantitatively studied using LDA type band structure theory. Computing T_c 's from first principles is very demanding and has only become possible recently by the increase of computer power. MgB_2 and a number of other cases have lend credibility to the case that these numbers can be trusted. This support the 'computational' case that 40K need not be the limit of conventional phonon driven superconductivity⁷.

Despite these caveats I am not aware of any expert believing that conventional phonons can be responsible for the superconductivity in cuprates or iron pnictides. Perhaps the best reason is that the extreme electron-phonon coupling required to explain a T_c of 150K should somehow reveal itself in band structure calculations, while these just indicate quite moderate couplings. This is not to say that phonons do not play any role. In fact, there are striking experimental indications that especially at very low doping the electron-phonon coupling is so strong that some variety of small polaron physics is at work. This appears to invoke polar couplings that are ignored in the LDA calculations where implicitly metallic screening is assumed. It cannot be excluded that even at optimal doping such a 'poor metal' polaronic physics is at work and claims that bipolarons are behind the superconductivity cannot be dismissed beforehand.⁸

The main stream thinks however differently. Staying within the "BCS paradigm" the quest to explain the high- T_c 's turns into the issue whether one can identify non-phonon stuff that can take the role of the phonons, mediating very strong attractive interactions. The only option on the table is to look for 'glue' that emerges from the interacting electron system itself. There are quite a number of options here, like the excitonic mechanism first proposed by Little, emphasizing special units in the crystal structure characterized by a high electronic polarizability.⁹ Another old idea is the Kohn-Luttinger overscreening effect in 2D systems¹⁰ and the closely related idea of the plasmon taking the role of glue. However, by far the most popular view has been up to the present day to point at spin fluctuations. The basic idea dates back to Berk and Schrieffer in the 1960's who aimed at explaining why metals that are close to a transition to an itinerant ferromagnetic state usually do not superconduct¹¹. They introduced a particular approximate way to compute matters that forms the conceptual core of the myriad of improved spin fluctuation theories that have been developed since then.

A first hurdle is that in the electron system at microscopic distances there is no such thing as a system of spins that has a separate existence of the electrons – the electrons just carry around the spins themselves. The 'spin glue' has therefore itself to be a highly cooperative phenomenon emerging at 'large' distances in the interacting electron system. Upon approaching a transition to a magnetic state in an itinerant metal one does however expect precursors of the magnetism in the paramagnetic state. These are in the form of short lived magnetic fluctuations that become increasingly better defined upon approaching the magnetic transition where they turn into real magnons – the paramagnons. Using time dependent mean field theory (RPA) the paramagnons can be computed perturbatively assuming weak interactions. Assuming in addition residual interactions between these spin fluctuations and the conduction electrons, their propagators can be just feeded into the Migdal-Eliashberg formalism to take the role of phonons.

The particular benefit of this approach is that it is quite predictive with regard to the pairing symmetries. For the case of ferromagnetic fluctuations Schrieffer and Berk showed that these generically cause repulsions in the s-wave channel explaining why nearly ferromagnetic metals usually do not

⁷See, e.g., Marvin Cohen's talk at the 2009 KITP superconductivity conference: <http://online.kitp.ucsb.edu/online/higherc09/cohen/>

⁸A.S. Alexandrov and N.F. Mott, *Polarons and Bipolarons*, World Scientific, Singapore (1995).

⁹This idea revived in the context of pnictides: see G.A. Sawatzky *et al.*, Eur. Phys. Lett. 86 (2009)174409.

¹⁰W. Kohn and J.M. Luttinger, Phys. Rev. Lett. 15 (1965) 524.

¹¹N.F. Berk and J.R. Schrieffer, Phys. Rev. Lett. 17 (1966) 433; an important follow up is S. Doniach and S. Engelsberg, Phys. Rev. Lett. 17 (1966) 750, focussing on the renormalization of the Fermi liquid.

superconduct. In the early 1970's superfluidity in fermionic ^3He was discovered. It was already known that the normal state is characterized by a strongly enhanced magnetic susceptibility indicating that the Helium Fermi-liquid is on the verge of becoming ferromagnetic. Given that there are no phonons around, the idea that this superfluidity had dealings with paramagnons was immediately realized. Dealing with phonons, the induced attractive interaction surely favors a spin singlet order parameter, while in good metals the effective coupling is localized in space and thereby momentum independent, favoring s-wave superconductivity. However, ferromagnetic fluctuations favor spin-triplet pairing. This in turn implies that the pairs need to be in an orbital state with uneven angular momentum, having the added benefit that the real space pair wave function has to have a node at the origin. The fermions in the pair thereby avoid each other automatically such that the hard core microscopic repulsions are circumvented. The bottom line is that although there might be a net repulsion between the quasiparticles this might well turn out to sum up into an attractive interaction in a higher angular momentum channel.

The superfluidity in ^3He is of the triplet variety and a discussion doing justice to the richness of this subject deserves a separate chapter in a book like this.¹² With regard to the microscopic pairing mechanism a worthwhile lesson is the use of the Berk-Schrieffer logic by Anderson and Brinkman to get with great effect the details of the A-phase right.¹³ The gross message is that ^3He demonstrates that an interacting fermion system as dominated by microscopic repulsive interactions can eventually turn into a superconductor, with the pairing mechanism having undoubtedly dealings with the spin fluctuations. However, the Anderson-Brinkman result is just showing that the spin fluctuations contribute to the attractive interactions, stabilizing a particular superconducting state, but it does not prove that the RPA style spin fluctuations do all the pairing work explaining in detail why T_c is what it is. For instance, much attention was paid as well to the role of the attractive tail of the microscopic Van der Waals interaction. Another example of this lack of microscopic understanding is the question whether the mass enhancements of the ^3He Fermi liquid can be accounted for by the coupling to the ferromagnetic spin fluctuations or whether they are related to the proximity of the crystallization transition¹⁴. It appears that the quest to explain the nature of the residual attractive force responsible for the pairing in ^3He ended in a stalemate. Many factors can contribute and it is just a number game. Given that there are no controlled mathematical methods that tell us how to connect the strongly interacting microscopy to the weakly interacting infrared, it is in fact plainly impossible to address this subtle quantitative matter with the available theoretical technologies. With an eye on the remainder, one should be aware that ^3He is in a way a best case scenario for glue ideas, since the normal state is a well developed Fermi-liquid, with a superfluidity that is clearly in the weak coupling limit, while complications like (lattice) Umklapp and phonons are absent.

The notion that the superconductivity of electron systems in solids could be caused by intrinsic electronic interactions effects got on the main stage for the first time by the discovery of the heavy fermion superconductors in the late 1970's and 1980's. The gigantic mass renormalizations in these systems leave no doubt that electron-electron interaction effects are dominant. In addition it became gradually clear that the superconducting order parameters in these system are unconventional by default while magnetism is usually around the corner. During the last twenty years a consensus emerged that in these systems the superconductivity is closely related to the presence of a quantum phase transition where at zero temperature a magnetically ordered state gets destroyed by quantum fluctuations. In section 2.4.6 I will take up this theme in detail, including a discussion of the Hertz type theories that are closely related to the spin fluctuations.

¹²The classic, hard to improve treatise is A.J. Leggett, Rev. Mod. Phys. 47 (1995) 331.

¹³P.W. Anderson and W.F. Brinkman, Phys.Rev.Lett. 30 (1973) 1108.

¹⁴D. Vollhardt, Rev. Mod. Phys. 56 (1984) 99; this in turn rested on the discovery by Maurice Rice and Bill Brinkman of the description of the Mott transition in terms of Gutzwiller projected wave functions: W.F. Brinkman and T.M. Rice, Phys. Rev. B2 (1970) 4302. These ideas were in turn inspirational in the development of the RVB type theories of section 2.4.4.

The spin fluctuation idea acquired directly prominence in the early history of cuprate high- T_c superconductivity. Obviously an agent different from phonons causing the superconductivity was at work, while in the frenzy of the late 1980's it became in no time clear that cuprate superconductivity emerged from doping an antiferromagnet. In addition, both inelastic neutron scattering- and NMR experiments gave away that the metals at superconducting doping concentrations show the signatures of pronounced spin fluctuations. The spin-fluctuation-glue idea was lying on the shelf and it was predictably taken up by part of the community. The most visible advocates were Douglas Scalapino and David Pines, representing somewhat complementary views that are in 2010 still discernible. Pines embodied the 'bottom-up' phenomenological approach, analyzing in great detail experimental information on the spin fluctuations, aiming for a quantitative phenomenological description of the superconductivity. Scalapino approached it from a more theoretical angle, trying to cast the (incomplete) information from quantum Monte Carlo computations and so forth in a diagrammatic framework tailored according to the Berk-Schrieffer framework and improvements thereof like the "fluctuation-exchange" (FLEX) perturbative theory.

In fact, the spin fluctuation advocates faced a heavy battle in the early history of high- T_c superconductivity. The predictive power with regard to the pairing symmetry was their stronghold. In the presence of Umklapp associated with the scattering of the electrons against the crystal lattice the pairing symmetry story becomes more interesting and specific. The lattice warps the electron dispersions and the Fermi-surface shape with the ramification that on the level of RPA calculations the spin fluctuations tend to get quite structured in momentum space. The cuprate Fermi-surface computed by LDA bandstructure (and confirmed by photoemission measurements) shows 'nesting' features that promote a strong enhancement of spin fluctuations at large momenta. This in turn implies a strong momentum dependence of the pairing interactions mediated by these fluctuations that strongly favor a $d_{x^2-y^2}$ superconducting order parameter. In the 1987-1990 era it was however taken for granted by the community at large that cuprate superconductivity was s-wave. In hindsight these strong beliefs are not easy to explain on rational grounds: the data were of a poor quality and it appears to be rooted instead in a conservative sociological reflex – even d-wave was too risky. After a hard fight, that lasted for a couple of years Scalapino and Pines *et al.* were proven right with their d-wave claim by the phase sensitive measurements of Tsui, Kirtley, van Harlingen and coworkers discussed elsewhere in this book. This is remarkable and I find it respectable that this success has motivated a stubborn attitude in the spin fluctuation community: there has to be something to the notion given that it got the d-wave right. However, after it became clear that the pairing symmetry is d-wave in no time it turned out that this is also the natural ground state in for instance the RVB context (section 2.4.4), and even invoking the spontaneous current phases of Varma as discussed in section 2.4.5. Although d-wave superconductivity is rather unnatural dealing with conventional phonons, the consensus has emerged that it is just too generic dealing with electronic mechanisms for it to be regarded as conclusive evidence for any specific mechanism.

At this moment in time the idea of a spin-fluctuation 'superglue' is very much alive – counting heads in the community it might be the most influential hypothesis all together. Part of this appeal is surely associated with the fact that it poses a crisp challenge to the experimentalists: the superglue is like a holy grail, and one just has to dig with the modern experimental machinery of condensed matter physics into the electron matter to isolate it. Fact of the matter is however that after twenty years of concerted effort in the cuprates the 'evidence' is as indirect as it has been all along. The state of the art is perhaps best represented by a tour de force that was published in 2009.¹⁵ Empirical information obtained from direct measurements of the spin fluctuations by neutron scattering and the electron self energy effects picked up in photoemission is combined with the results from numerical calculations, arriving at the claim that the spin fluctuation glue can fully account for the superconductivity. But this work also reveals the weaknesses of this 'paradigm'. The evidences are

¹⁵T. Dahm *et al.*, Nature Physics 5 (2009) 217.

highly quantitative, while it departs from oversimplified toy models where one nevertheless loses mathematical control because of the intermediate strength of the couplings involved. Equally convincing cases have appeared arguing that the self energy kinks in photoemission are entirely due to rather weak electron-phonon coupling. The magnetic resonance that dominates the spin fluctuation spectrum in optimally doped cuprates can be interpreted rather as an effect than as the cause of the superconductivity. Perhaps the most devastating counter evidence available at present for such an interpretation is the staggering 'electronic complexity' of the underdoped pseudogap regime as discussed in section 2.4.5. The simple fermiology 'RPA view' on this competing order appears as a gross oversimplification of this reality.

The advent of the pnictide superconductors appeared initially as a shot in the arm for the spin fluctuation idea. The LDA calculations indicate nesting properties of the Fermi-surface pockets that are beneficial for the RPA paramagnons, while they are consistent with the " $(\pi, 0)$ " magnetism found in the parent compounds. The case was made vigorously that since the parent systems are not Mott insulating it has to be that the iron superconductors are quite itinerant and typical candidates for spin fluctuation superconductivity. Last but not least, a peculiar " s_{\pm} " pairing symmetry was predicted based on the LDA-RPA-Eliashberg "trinity".¹⁶ It is now widely believed that this symmetry is indeed realized in at least some of these systems, although definitive evidence is still missing.

Much of the early experimental activity in this young field was aimed at trying to find evidence for the spin-fluctuation superglue. Remarkably, when the moment seemed fit to cry victory the building started to crumble. Data appeared indicating that unexpected 'nematic' phenomena are happening in the underdoped pnictides that seem to parallel the pseudogap complexities of the cuprates. In addition, the pnictide metals at optimal doping have weird properties like a linear resistivity that again suggests that these have uncanny commonalities with the mystery stuff of the cuprates. This is a very fresh development and it is much too early to say where it will land. However, I do expect that it is beneficial for the newcomers in the pnictide field to be aware of the very long and contentious history of the spin fluctuation glue idea.

1.3 The legacy of Philip W. Anderson (I): Mottness.

When in some future the real history of T_c can be written, the odds are high that Philip Anderson will have a legendary role to play. The influential Andersonian tradition in high- T_c superconductivity revolves around the assertion that the ground rules of metal physics as laid down in the 1950's do not apply to the electrons in cuprates, for the reason that high- T_c superconductivity emerges from a Mott insulator that is doped. This is in stark contrast with the spin-fluctuation school of thought as discussed in the previous section. This departs from the notion that one can get away with the idea that the normal state is in essence a Fermi-liquid. Although shaken hard by the powerful glue bosons, the quasiparticles forming a Fermi-gas are eventually ruling the waves and it is just a matter of sorting out the Feynman graphs governing the perturbations of the quasiparticle gas. When high- T_c was discovered in 1986 this was taken as a self-evident truism by nearly everybody. However, Anderson published one of the first papers ever on the subject and this is still the most influential paper in the history of the subject introducing the 'resonating valence bond' (RVB) idea¹⁷ that will be the focus point of the next section. This section is a preliminary: the success story of nailing down the specifics of the parent cuprate Mott insulator in the late 1980's.

¹⁶This appears to have been a close race, with winners: I.I. Mazin *et al.*, Phys. Rev. Lett. 101 (2008) 057003, being cited already 500 times.

¹⁷P.W. Anderson, Science 235 (1987) 1196.

Mott insulators are in fact much simpler than band insulators: they are just the incarnation of rush hour traffic in the world of electrons. The Coulomb repulsion dominates with the effect that electrons have to avoid each other. When only one electron fits every unit cell, while there are as many unit cells as electrons all 'electron traffic comes to a complete stand still'. Doping has the effect of making some room for motions – it is the hbar version of stop and go traffic. The first fact that got settled in high- T_c superconductivity was the Mott insulating nature of the undoped cuprate parents.

This matter actually jump started my own career. I was quite lucky having George Sawatzky as thesis supervisor. In the early 1980's photoemission was still quite young and George had the vision to use this technique to focus in on the classic problem of the electronic structure of insulating transition metal salts. This was also the era that the local density approximation (LDA) band structure description of the electronic structure was at the height of its success, having demonstrated that it delivered a remarkably accurate description of the band structures of conventional metals and insulators. Unaware of its contentious history, the claim was made by band structure theorists in 1984 that straightforward spin dependent LDA did describe the electronic structures of 3d transition metal oxides correctly. This triggered a ferocious response by Anderson. I have a vivid recollection reading a short paper he wrote that never got published, written with a literary quality that is rare in the physics literature: "Williams et al. arrive at the laughable contention that cobalt oxide is a metal because their band structure calculation is telling so."

Apparently this response was rooted in developments in the 1950's. Slater might be viewed as the father of modern band structure theory. With his $X - \alpha$ method he had outlined the idea of effective exchange-correlation mean-field potentials and one can view the later work of Kohn and so forth as an a-posteriori formal justification of the basic notion. Apparently Slater believed that his mean-field potentials were the last word and it is rumored that Slater gave the young Anderson a very hard time.

In this era Mott and van Vleck, Anderson's intellectual fathers, had discovered the basics of what we nowadays understand as 'strongly correlated electron physics'. This revolved initially around the notion that the electron physics in transition metal salts is dominated by strong, local electron repulsions. Although Hubbard can be credited to be the first who wrote down the Hubbard model explicitly in 1964, Mott introduced the notion of Mott insulator in 1956, while Van Vleck already had in the 1940's a clear notion of the local Coulomb interaction (U) competing with the hopping t ¹⁸. Anderson himself has a big stake in this early development in the form of his 1959 work on the superexchange mechanism for the spin interactions in Mott insulators¹⁹, as well as his Anderson impurity problem (awarded with the Nobel prize), representing landmarks of the physics of strongly correlated electron systems.

In the period where the band structure/fermiology view was turning into the main stream the solid state physics community was not particularly receptive to the Mott-insulator bad news. When this old struggle revived in 1984, we were making in Groningen good progress, finding interesting signals of the 'Mottness' at work in transition metal salts in the form of satellites and so forth in various electron spectroscopies. Together with Jim Allen, Sawatzky realized that this new information could be used to make a difference in the old debate and this culminated in the 'Zaanen-Sawatzky-Allen' theory for the electronic structure of Mott-insulating salts. We explained how the essence of the toy models designed by Anderson and Hubbard could be combined with the complexities of the electronic structure of the real salts. In the first half of the 1980's nobody seemed interested, but this changed drastically when high- T_c superconductivity was discovered. Cuprates were an important part of the ZSA ploy as archetypical 'charge transfer insulators' where the holes of the p-type

¹⁸According to Phil Anderson, Van Vleck used to call it the UT model, a typical side reference to the University Theater, Harvard Square ...

¹⁹P.W. Anderson, Phys. Rev. 115 (1959) 2.

cuprates are actually formed from oxygen states. My thesis work was to a degree rediscovered by others: Emery with his 'Emery model' and Zhang and Rice with their 'singlet' (Sawatzky's idea of starting with 3d impurities). Rather quickly this converged into a main stream view, insisting that one should be able to get away in the doped systems with the minimal t-J model, much in the spirit of Anderson's RVB philosophy. The marked exception was Chandra Varma. He was convinced that 'chemistry matters', that one has to worry about holes being on oxygen. After a lonesome but resilient search of some twenty years this triumphed recently, in the form of Varma's intra-unit cell spontaneous currents, as will be discussed in section 2.4.5.

In the years that followed a massive effort ensued aimed at understanding the parent Mott insulators. The experimental- and theoretical technologies were just lying ready to be used to completely clarify their physics. It appears that this resulted in a seemingly complete understanding of Mott-insulators. We have learned how to modify LDA so that it can cope quantitatively with the real life Mott insulators through the LDA+U and LDA+DMFT functionals. A detailed understanding has been achieved of the physics of an isolated carrier moving through the antiferromagnet resting on linear spin-wave theory and the self-consistent Born approximation, with the recent addition that in the insulators the polar electron-phonon couplings are actually quite important. The study of Mott-insulators has turned in the mean time into a form of standard science in material-science laboratories, but unfortunately this has not made much of a difference with regard to the still highly mysterious physics of Mott-insulators turning into metals and superconductors when doped.

1.4 The legacy of Philip W. Anderson (II): Resonating Valence Bonds and its descendants.

Upon learning in 1986 that high- T_c superconductivity did occur in a doped copper-oxide, Anderson directly arrived at the hypothesis that the phenomena should be rooted in the 'strangeness' of the electron states subjected to the "Mottness". His RVB paper heralded a remarkably intense and creative period in condensed matter theory. The 'spins and holes' ideas that are still at the center of the thinking regarding unconventional mechanisms were laid down in a high pace in the years 1987-1988, under the intellectual leadership of Anderson. The emphasis is here on 'ideas': it actually amounts to inspired guesswork, not disciplined by controlled mathematics, while until the present day empirical evidence for 'RVB-ish' physics is quite thin. With the exception of the notion that Mottness is changing the ground rules, my impression is that even Anderson will admit that the problem of high- T_c superconductivity is still wide open. However, regardless whether they apply literally to cuprate superconductors, the ideas that emerged in 1987/1988 are embodying a very interesting theoretical framework that deserves to be realized by nature somewhere.

The essence of the Mott-insulator is that the electrons that should form a high density degenerate Fermi-gas, turn instead into a system of spins due the domination of the local Coulomb repulsions. This is simple to understand, but the consequences are remarkable. The free Fermi-gas is in first instance governed by Fermi-Dirac statistics, rooted in the fact that electrons are indistinguishable fermions. Due to the interactions the electrons localize, thereby becoming distinguishable particles, and all what is left behind are their spins that live as distinguishable degrees of freedom in tensor product space. Upon doping, the moving holes release 'here and there' the indistinguishable nature of the electrons. In the doped Mott insulator the electrons are most of the time distinguishable spins, but once in a while when a hole passes by they remember that they are actually fermions. At stake is that 'Mottness' interferes with the fundamental rules of quantum statistics. This is a rather recent insight – I learned it from Zheng-Yu Weng, and it appears that it is not quite appreciated yet by the community at large. Whatever it means, I find it myself a particularly helpful point of departure to

appreciate why RVB and so forth makes sense.²⁰

The point of departure of Anderson's seminal RVB paper is the idea that the pairing of electrons is already imprinted in the spin system formed in the Mott-insulator. The common phenomenon in these Heisenberg spin systems is that the spins order in simple antiferromagnets. After the dust settled in 1987/1988 this turned out also to be the case in the high- T_c parents. However, in two dimensions and dealing with small $S = 1/2$ spins there is a potentiality that the quantum fluctuations become so strong that some quantum spin liquid is formed. Anderson envisaged such a liquid, actually inspired by old ideas of Pauling. Configurations from pairs of spins forming singlets ("valence bonds") are formed, and the spin liquid corresponds with coherent superpositions ("resonating") of all possible tilings of the lattice with such singlet configurations. Upon doping, the electrons forming the singlet pairs become mobile, and the valence bonds turn into Cooper pairs that condense in a superconductor.

The idea is simple and appealing but the hard part is to demonstrate that the doped spin system is actually dominated by these pair-wise singlet configurations. After all these years definitive evidence for such a state is still missing. There is no controlled mathematical- or computational procedure available for the doped Mott-insulator problem, as rooted in the statistical troubles alluded to in the above ('fermion signs'). At the same time, there is no experimental machinery available that can measure the RVB 'entanglement' directly and the evidences that are quoted to support RVB type physics are invariably indirect – the pseudogaps, d-wave superconductivity and so forth can be as well explained within e.g. the conventional spin fluctuation view.

The 'idea revolution' that ensued in the period 1987/1988 was driven by the desire to pack mathematical meat on the conceptual bones of Anderson's RVB idea. Anderson took a first step in this direction in his 1987 Science paper by introducing the idea of a particular kind of variational wave-functions. This dates back to the helium era where Jastrow introduced the idea of Ansatz wavefunctions that are derived from the free Fermi/Bose gas but where one wires in that the particles avoid each other locally due to strong local repulsions.²¹ Gutzwiller suggested a special form of such Ansatz wavefunctions, just amounting to a free fermion ground state where the double occupied states are projected out.²² These were in 1972 applied by Maurice Rice and Bill Brinkman, in the context of the Mott metal-insulator problem. Anderson pointed out in his RVB paper that projecting out double occupancy from a free BCS ground state wavefunction (instead of the Fermi gas) yields a state which is just like his envisaged RVB superconductor. To do this well one has to evaluate matters numerically, and this variational Monte-Carlo approach turned into a main stream with hundreds of papers published.

Despite strong claims, this numerical activity has not been decisive up to the present day. The Gutzwiller style variational approaches suffer intrinsically from the 'garbage in-garbage out' problem. One wires in through the Ansatz a particular prejudice regarding the physics and it is hard to judge from the outcomes whether this was a good idea, given the well established wisdom that ground state energies and so forth are not very sensitive measures of the correctness of the wave function. In fact, the failure of the computational methods at large can be viewed as the best evidence for the profundity of the doped Mott insulator problem. All numerical methods characterized by a real measure of control show that they eventually loose this control. Quantum Monte Carlo and the closely related high temperature expansions crash before temperature is low enough for matters to become truly significant because of the sign problem. The density matrix renormalization group fails in two (and higher) dimensions by its inability to keep track of the quantum entanglements. Dynamical mean field theory (DMFT) is based on the uncontrolled ad-hoc assumption that the ef-

²⁰see J. Zaanen and B. Overbosch, arXiv:0911.4070. See also P. Phillips. Rev. Mod. Phys. Coll. 82 (2010) 1719 for a complimentary perspective on Mottness.

²¹R. Jastrow, Phys. Rev. 98 (1955) 1479.

²²M.C. Gutzwiller, Phys. Rev. Lett. 10 (1963) 159.

fects of electron correlation effects are purely local and in this regard it is not that much different from a primitive Gutzwiller Ansatz. Right now perhaps the most promising approach is the 'dynamical cluster approximation' as designed by Jarrell that can be viewed either as a finite cluster Quantum Monte Carlo with cleverly designed effective medium boundary conditions, or either as a finite range extension of the DMFT. At the least this approach has delivered a real surprise in the form of a phase separation type quantum phase transition, happening in the Hubbard model at intermediate coupling.²³

The theoretical pursuit triggered by the RVB idea eventually settled in what is called the 'gauge theories of strongly correlated electron systems'. Several motives that were around got combined into a new view on the collective behaviors that at least in principle can exist in strongly interacting quantum matter. Perhaps the most important motive is the notion of quantum number fractionalization: the idea that the strongly interacting, highly collective state of matter might carry excitations that behave like weakly interacting particles/elementary excitations carrying however 'fractions' of the quantum numbers of the elementary particles. In the context of the RVB school of thought this turned specifically into the notion that the electron quasiparticle of Fermi-liquid theory 'falls apart' in particles that carry separately spin 1/2 ('spinons') and electrical charge ('holons'). The fractionalization 'phenomenon' was first identified in the 1970's by Jackiw and Rebbi in the mathematical study of field theories in one space dimensions (Thirring-, Sine-Gordon models) while it acquired great fame in condensed matter physics when it was rediscovered by Su, Schrieffer and Heeger in the simple, physical setting of the excitations of polyacetylene.²⁴

This is just a dimerized (double bond-single bond) chain of carbon atoms and 'SSH' recognized that a kink/domain wall in this dimerization pattern carries an electronic bound state that is formed from half a conduction band- and half a valence band state. The consequence is that it corresponds with a pure $S=1/2$ excitation carrying no charge ('spinon'). When one binds a carrier to this soliton it turns into a spin-less excitation carrying the electron charge ('holon'). Kivelson, Rokhsar and Sethna recognized that a 'short range' RVB state formed from nearest-neighbor singlet pairs (described by 'quantum dimer' models) is for the purpose of fractionalization just a two dimensional extension of the SSH physics.²⁵ As long as the dimers form a liquid it is simple to see that isolated spins and -holes are sharply defined topological excitations that carry either spin 1/2 or charge. Fradkin, Kivelson and others soon thereafter managed to link the quantum dimer models to gauge theory.²⁶ They showed that it is dual to a version of compact quantum electrodynamics in two space dimensions, linking the fractionalization to the confinement-deconfinement phenomenon, as known from e.g. quantum chromodynamics.

However, compared to the high energy physics context the perspective is warped in a way that continuous to confuse high energy theorists. The lore in fundamental physics is to consider the quarks as the fundamental particles and due to the anti-screening property of the gluon fields, these are confined at low energy in baryons. The rules precisely reverse in the quantum dimer model: the electron is the fundamental object, but it only exists in the confining state of the theory. The spinons and the holons have the status of the quarks, but they appear as highly collective excitations rooted in the global properties of the 'topologically ordered' RVB liquid. These only become real when the ground state/vacuum of the gauge theory is deconfining. Therefore, deconfinement is a profound emergence phenomenon while the confining state is just about no-brainer microscopic 'chemistry' surviving up to the longest distances. This interesting motive is shared by all theories dealing with the 'emergent gauge principle' in condensed matter physics.

The quantum dimer problem on the square lattice actually died in a rather interesting way. Com-

²³see E. Khatami *et al.*, Phys. Rev. B81 (2010) 201101.

²⁴see the classic review: A.J. Heeger, S. Kivelson, J.R. Schrieffer and W.P. Su, Rev. Mod. Phys. 60 (1988) 781.

²⁵S.A. Kivelson, D.S. Rokhsar and J.P. Sethna, Phys. Rev. B35 (1987) 8865.

²⁶See Eduardo Fradkin's book *Field theories of Condensed Matter Systems*, Addison-Wesley (1991).

pact pure gauge electrodynamics is always confining in 2+1D. In the 'dual' quantum dimer model context this translates into the fact that at least on a square lattice the ground states always correspond with valence bond crystals where the valence bonds just stack in a regular manner breaking the lattice translations. This has the consequence that spinons and holons are confined, since they are tied together by domain wall 'strings' in the valence bond crystals. Much later Moessner and Sondhi discovered a loophole, by their demonstration that the dimer model of the 2D triangular lattice does support a deconfining state.²⁷ This is perhaps still the best proof of principle that fractionalization as linked to the notion of deconfinement in gauge theories does make sense, at least theoretically.

Another, at first view unrelated way, that the gauge principle enters was recognized by Baskaran and Anderson, now directly associated with the very nature of Mottness itself. This relates directly to the 'stay at home' principle that emerges at low energy in the Mott-insulator. The Mott-localization implies that in fact the *local* number operator is sharply quantized: in the Mott-insulator one can count precisely the (integer) number of electrons within a finite localization volume. Baskaran and Anderson argued that one can work with free fermions as long as one couples these to a compact U(1) gauge field that in turn imposes the local conservation of charge.²⁸ Affleck and Marston took this one step further by noticing that the Heisenberg type spin Hamiltonian is in turn also invariant under charge conjugation (it doesn't matter whether one describes spins in terms of electrons or holes), showing that together with the 'stay at home' gauge the overall symmetry of the gauge fields in the Mott insulator is SU(2).²⁹

The final crucial motive that was required for the mathematical underpinning of the gauge theories for doped Mott-insulators is the idea of the large N limit. N refers here to the number of degrees of freedom in a field theory. The notion that one can organize matters in terms of a perturbation expansion where $1/N$ acts as the small quantity dates back to the hay days of non-abelian Yang-Mills theory in the 1970's. The Dutch theorist 't Hooft realized that the diagrammatic perturbation theory of the pure gauge $SU(N)$ Yang-Mills theory drastically simplifies in the limit that $N \rightarrow \infty$, since one only has to consider the so-called planar diagrams. When the gauge coupling is strong (the interesting- and physically relevant case) one still has to re-sum an infinity of diagrams but these now form dense two dimensional "nets" in space time that are topologically trivial (no handles). This motive evolved in a crucial conceptual pillar underpinning the gauge-gravity dualities of modern string theory: these planar diagram networks can be viewed as string worldsheets that are topologically trivial, with the ramification that the large N limit of the gauge theory behaves like string theory at small string couplings.

The way that large N enters in the context of strongly interacting electron systems is actually very different, obviously so because there is no such thing as fundamental non-abelian gauge bosons at the microscopic scale. This involved a leap of imagination by Ramakrishnan and Sur,³⁰ who showed that the large degeneracy of the strongly interacting 4f atomic states in the valence-fluctuating/heavy fermion intermetallics could be exploited for the purposes of a large N expansion. It soon became clear that the Kondo- and Anderson impurity problems drastically simplify in the limit of high orbital degeneracy. A next crucial step was taken by S. Barnes, followed soon thereafter by Newns and Read with the discovery that 'slave' mean field theories control the physics at large N .³¹

This construction can be considered as the central wheel in the 'modern' formulation of the

²⁷R. Moessner and S.L. Sondhi, Phys. Rev. Lett. 86 (2001) 1881; see also A. Kitaev's honeycomb model, Ann. Phys. 321, (2006) 2.

²⁸G. Baskaran and P.W. Anderson, Phys. Rev. B37 (1988) 580.

²⁹I. Affleck *et al.*, Phys. Rev. B38 (1988) 745.

³⁰T.V. Ramakrishnan and K. Sur, Phys. Rev. B26 (1982) 1798.

³¹See D.M. Newns and N. Read, Adv. Phys. 36 (1987) 799 for an account of this early era of slave theories. Piers Coleman deserves special mention for his work on the impurity models, apparently be the father of the idea of slave particles: Phys. Rev. B 29 (1984) 3035.

emergent gauge principle in strongly interacting fermion matter. It demonstrates the potentiality of gauge symmetry- and dynamics as a controlling principle for the emergent highly collective physics at long distances and times, in a manner that is completely different from the way the gauge principle is understood 'as coming from god' in high energy physics. At first sight the construction seems absurd. One starts out with for instance an Anderson lattice model describing a lattice of strongly interacting f-shell electrons hybridizing with a non-interacting valence band, such that one of the two low energy valence states of the f-shell has a large orbital degeneracy N . One then asserts that the f-electron field operators can be written as products of creation/annihilation operators of two particles, one carrying the spin- ('spinon') and the other the charge ('holon') of this electron. The electron is a fermion and therefore one has to make a choice how to divide the quantum statistics over the slave particles. One can either associate a bosonic statistic to the holon such that the spinons are fermions ('slave fermions'), or the other way around ('slave bosons'), while pending the choice one ends up with quite different theories. Surely these slave particles are unreal: they correspond with redundant (gauge) degrees of freedom that can be removed by imposing gauge invariance invoking explicit gauge fields. But then one discovers that in the large N limit the local constraints that tie the spinons and holons together turn into simple global chemical potentials, which is the same statement that the gauge couplings disappear. The consequence is that the effective gauge theory is in a deconfining regime such that the spinons and holons become as real as quarks above the confinement transition!

One is not done yet because the resulting theory in terms of 'physical' spinons and holons has no free part: it only contains two particle (spinon-spinon, spinon-holon) interaction terms. But these can be handled in terms of mean-field theory, introducing single boson- (Bose condensation) and bilinear fermion vacuum expectation values. In the original context of Anderson lattice models this was used to explain the emergence of the heavy electron bands. However, soon after the RVB ideas of Anderson had sunk in a variety of theorists realized more or less independently that this large N slave theory route seemed quite promising with regard to explaining high- T_c superconductivity when applied to the appropriate large N generalizations of the t-J model. In case that the fermion statistics is attached to the spinons these mean field theories are quite like the Gutzwiller type theories for RVB states, while they also include the fractionalized excitations suggested by the quantum dimer theories. The spin vacuum is build up from pair singlets with a hard-wired d-wave type coherence, described in terms of a BCS-like mean field structure at 'infinite' coupling. The doped holes turn into holons, that Bose condense directly giving rise to a 'holon superfluid' with a superfluid density that increases with the number of doped carriers. At the same time, the spinon BCS state has a T_c that decreases with doping. From gauge invariance it is easy to understand that one has a physical superconductivity only when both holons and spinons are condensed, explaining the doping dependence of the superconductivity. Above all, it becomes trivial to explain why T_c is high because the scale is set by the large superexchange interaction J . The underdoped state at higher temperatures where the spinons are condensed, but the holons are still normal is suggestive of the pseudo-gap regime, while it can also be argued that the (overdoped) state where the spinons form a metal while the holons are condensed should eventually correspond with a Fermi-liquid.

The benefit slave theories in the large N limit surely lies in the sense of mathematical control while the physical implications are remarkable.³² The headache is however that N in the physical case is only two (spin degeneracy) and it is highly questionably whether the large N limit has anything to say about the real system. For any finite N the gauge couplings in the bare theory jump to infinite since Maxwell terms for the gauge fields are non-existent. This would mean that one lands in a strongly confining regime, where the spinons and holons bind in the original electrons such that nothing is achieved. But one can then argue that short distance perturbative corrections involving the spinons and holons will generate dynamically a gauge kinetic energy. This might

³²For a recent in depth review see, P.A. Lee, N. Nagaosa and X.-G. Wen, Rev. Mod. Phys. 78 (2006) 17.

then stabilize a deconfining state, but the next problem is that one has to find reasons that in the two space dimensions of the cuprates the monopoles associated with the compactness of the gauge field can be kept out of the vacuum. This gave rise to much debate over the last twenty years and a consensus emerged that in principle even in two dimensions the deconfining state has a chance to exist in principle.

The slave particle gauge theory is up to the present day highly contentious as an explanation for superconductivity at a high temperature, and not only because of the lack of real mathematical control in the physically relevant small N regime. The greater problem is that despite many attempts this theory has failed to produce striking insights and predictions for experiment. It offers suggestive explanations for a number of observations, but these can also be explained in a multitude of different ways (like the superconducting dome, the pseudogap, and so forth). In order to be taken seriously, a theory that departs as radically from established wisdoms as these gauge theories, it should produce 'smoking gun' predictions that can be tested by experiment. After more than twenty years, such hard empirical evidence is still not available.

Another matter is that this development in high- T_c context has inspired a revival of the field of quantum magnetism. Pure spin systems tend to order "classically" in the form of (anti)ferromagnets. When the microscopic spins have a small magnitude (like the $S=1/2$ spins of the cuprates) the quantum fluctuations around this classical order can be quite serious. The challenge is to find out whether quantum spin systems exist that do not order in a conventional magnet but instead have a 'quantum spin liquid' ground state. This is a natural arena of the slave particle gauge theories discussed in the above paragraphs. There are a number of indications that the slave theories are landing on the real axis in this context. An early success is the spinon theory in bosonic incarnation by Arovas and Auerbach.³³ This representation is especially suited to track the physics of quantum spin systems that fail 'at the last minute' to turn into classical order: it turned out to be very closely related to the renormalization group for the quantum non linear sigma model discussed in the last section. A next success was the large N slave fermion spinon theory in the CP_N incarnation by Sachdev and Read³⁴, predicting the valence bond crystals in two space dimensions. It appears that in the mean time this state is now well established in the $J_1 - J_2$ frustrated Heisenberg spin systems³⁵, while the case is compelling that this valence bond 'crystallization' is playing an important role in the stripe physics discussed in the next section. The key question is however whether *compressible* quantum spin liquids exists, systems that show no manifest classical order while their excitation spectrum is gapless. Believing the fermion slave theories such states should be generic, corresponding with states where the fermions form effective Fermi-liquid systems characterized either by large Fermi-surfaces or states that look like unconventional superconductors with massless nodal excitations. In fact, in this context the most fanciful mathematical considerations were formulated, in the form of the classification of spin liquids in terms of the projective symmetry groups by X.-G. Wen.³⁶ This work exposes more than anything else the unusual nature of the symmetry at work in the 'slave worlds'. The essence of Wen's construction is the recognition that the gauge volume of the slave theories is not spanned by separate gauge fields but instead by gauge equivalent slave mean field ground states. Wen derives the precise symmetry algebra's for this 'projective symmetry', and shows how to derive the elementary excitations in Goldstone theorem style, coming up with a zoo of compressible 'fermion like' spin liquids.

Quite recently an organic system was discovered in Japan where the empirical evidences are quite serious for the existence of such a 'spinon Fermi-liquid'. This system forms a geometrically frustrated two dimensional lattice of organic molecules containing one electron per site, forming a Mott

³³D.P. Arovas and A. Auerbach, Phys. Rev. B38 (1988) 361.

³⁴N. Read and S. Sachdev, Phys. Rev. B42 (1990) 4568.

³⁵This got much attention in the entertaining context of 'deconfined quantum criticality', T. Senthil *et al.*, Science 303 (2004) 5663.

³⁶This was introduced in: X.G. Wen, Phys. Rev. B65 (2002) 165113.

insulator that is very close to the transition to a metal. Applying a small pressure it undergoes a first order transition into a metallic state that suggestively turns into an unconventional superconductor at low temperatures. The insulating magnet does not show any hint of long range order down to the lowest temperatures while several properties are quite suggestive of a "spinon" Fermi-liquid type of physics.³⁷

1.5 Theory that works: the competing orders of the pseudogap regime.

In the cuprates the emphasis shifted in the last decade or so to the investigation of the underdoped pseudogap phase. The reason is that high T_c superconductivity is a very empirical branch of physics and there is much to explore in the pseudo gap phase using the modern machines of condensed matter physics. This has developed into a kind of embarrassment of the riches. It is now clear that competing order is at work, in tune with the idea of a quantum phase transition at optimal doping (next section), but the evidence points actually at a whole collection of exotic ordering phenomena. Not so long ago there was much disagreement among the different sub-communities focussed on particular orders, but a consensus emerged that these are all real. The present state of affairs was anticipated a number of years ago by Bob Laughlin, who described this situation in his popular book³⁸ as the 'dark side of emergence', arguing that the strongly interacting electron matter of the cuprates is complex to a degree that approaches biology. The ramification would then be that theoretical physics is side-tracked dealing with such complexity. Although Laughlin's prophecy regarding the pseudogap phase turned out to be quite accurate, I do find his view overly pessimistic. In the first place, when one heats up the pseudogap "stuff" one reenters the "marginal Fermi-liquid" normal state at some higher temperature. As I will discuss in the next section, this is by itself a monument of simplicity that deserves to be explained by equations of an Einstein-Dirac quality. The significance of the pseudo-gap 'complexity' then lies in the fact that apparently the simple quantum critical metal of the cuprates carries the *potentiality* to "fall apart" in a variety of exotic ordered states. Andy Mackenzie gives it a wonderful engineering twist: the stuff that makes up the quantum critical metals of the next section corresponds with construction material that makes possible to build states of quantum matter that cannot be constructed from stable states like the Fermi liquid.

Laughlin gets it really wrong with regard to the role of theorists. Although my guild has failed miserably with regard to impacting on the understanding of the disordered quantum liquid states, theory has been remarkably successful in guiding the experimentalists in discovering the exotic orders at work in the pseudogap phase. Apparently the mathematical machinery that is lying on the theoretical physics shelf is particularly geared towards understanding order in the sense of symmetry breaking.

Among the phenomena that have been identified in the pseudogap phase, only the notion of low superfluid density, strongly phase fluctuating superconductivity was in principle understood before 1987. The naive version of this is the idea of 'preformed pairs' but the reality in cuprates is better captured by Laughlin's metaphorical 'gossamer superconductivity'.³⁹ The phase fluctuations of the superconductor are caused by the struggle for domination of the superconductivity with the com-

³⁷See P.A. Lee, Science 321 (2008) 1306.

³⁸R.B. Laughlin, *A different Universe: Reinventing Physics from the Bottom Down*, Basic books (2005).

³⁹Given that the concept of phase fluctuating superconductivity is quite conventional, the literature is large. It starts with several groups in the late 1980's that are largely ignored because they were ahead of the fashion. At least counting citations, V.J. Emery and S.A. Kivelson, Nature 374 (1995) 434 has been perhaps most influential. Laughlin stresses the proximity to the Mott insulator as a perspective to understand the 'thin superconductor': R.B. Laughlin, Phil. Mag. B86 (2006) 1165.

peting states. The proximate Mott-insulator that hangs like a big thundercloud over the pseudogap regime is in this regard a major factor (the 'gossamer' idea).

The other 'exotic' pseudogap orders that have been identified in the mean time were invariably not known in 1987. These were first discovered theoretically, motivating experimentalists to look for them in less obvious corners. These gained general acceptance only after much controversy: testimony for the novelty involved in the discoveries. In historical order this is about the stripes, the quantum liquid crystal orders and the intra-unit cell spontaneous diamagnetic currents.

The stripes were the first to be discovered and they have at present the status of best established among the exotic pseudogap orders. They came into existence on some Friday afternoon in early October 1987, when I was fooling around with a computer code that produced a very unexpected output.⁴⁰ This landed on the real axis by the observation of static stripes with neutron scattering in the so-called 214 LTT cuprates by Tranquada *et al.* in 1995.⁴¹ while it took until this century for it to become fashionable. At another occasion I will give a detailed account of this long history. The bottom line is that I got myself eventually lavishly rewarded for my resilience, but I also learned to be quite cynical regarding the agility and open mindedness of the relevant part of the physics community.

What are the stripes about? When it started in the 1987 it was perhaps in first instance a symbol for the idea that the electron matter in cuprates could be more organized than in conventional metals. The resistance against the idea that the electron stuff could be 'inhomogeneous' is just rooted in the way hard condensed matter physicists are trained. Much emphasis is put on the physics of Fermi- and Bose gasses that make much sense dealing with the kinetic energy dominated electron matter in conventional metals. However, dealing with doped Mott-insulator, rush hour traffic is a more useful metaphor: when the insulator is like the complete jam, stripes are like the dynamical patterns that occur when the electron traffic enters the stop- and go regime of the lightly doped insulator⁴² Early on, there were more theorists playing around with the inhomogeneity motive, including Lev Gorkov, as well as Vic Emery and Steve Kivelson who were pursuing in the late 1980's the idea of frustrated phase separation droplets.⁴³

Stripes are however a quite specific form of organization that is directly tied to the proximity of the Mott insulator. The carriers form lines ('rivers of charge') in the two dimensional cuprate planes, separated by Mott-insulating 'stripes' that support antiferromagnetic order, with the specialty that the 'rivers of charge' have a double role as domain walls in the antiferromagnetism. From the point of view of symmetry breaking these correspond just with a conventional electron charge density wave coexisting with an incommensurate colinear antiferromagnetic order ('spin density wave'). However, the microscopic physics is quite special for the doped Mott insulator. The main contribution of Olle Gunnarsson to the first paper was his vision that the stripes should be viewed as the electronic version of a *discommensuration lattice* – we borrowed the designation 'stripes' from the naming of such states as they occur in classical incommensurate systems. Hubbard and t-J models are just crude toy models, introduced to capture some essence of the physics. Viewed generally, Mott insulators are electron crystals that are commensurate with the underlying ionic lattice, and the Mott gap is nothing else than a 'commensurate pinning energy' causing a gap in the acoustic

⁴⁰For the record, this work was presented for the first time in a seminar in december 1987 at the ETH Zürich, and in a large oral session at the Interlaken M2S meeting in Februari 1988. The paper was submitted in March 1988 to Physical Review Letters and after 1.5 years it was transferred to Physical Review B while the original submission date was removed: J. Zaanen and O. Gunnarsson, Phys. Rev. B 40 (1989) 7391. I kept the files with referee reports – it is a hilarious read. A couple of months after submission I got an e-mail by Machida explaining that he had seen our preprint while they had independently discovered the same motive. All other papers in this early era came much later. Especially, the late Heinz Schulz approached me after my Interlaken talk explaining that he found it very interesting, planning to take up the theme by himself. Apparently, back in Paris he had forgotten where he got the idea.

⁴¹J.M. Tranquada *et al.*, Nature 375 (1995) 561.

⁴²See J. Zaanen, Science 315 (2007) 1372 for a further elaboration of this metaphor.

⁴³V.J. Emery and S.A. Kivelson, Physica C209 (1993) 597.

phonon spectrum of the electron crystal. Upon doping, the lattice constant of the electron crystal becomes incommensurate relative to the ion lattice and since the pinning is strong the mismatch gets concentrated in small areas (discommensurations, the charge stripes) while elsewhere the electron crystal stays commensurate (the Mott insulating domains). A difference with the classical incommensurate systems is the spin degeneracy. This is lifted by the short range quantum fluctuations turning the spin system into an incommensurate antiferromagnet. My 1987 computation was just rooted in Hartree-Fock ('classical saddle point', 'old fashioned mean field theory', whatever) and dealing with classical orders (like antiferromagnets, electron crystals) the power of this method is at least in qualitative regards remarkable. It got the stripes quite right.

Stripes are now well known to be ubiquitous in doped Mott-insulators. Usually (nickelates, manganites) they submit however to the Hartree-Fock stability rule that they should keep the doped Mott insulator insulating. The cuprates are in fact the exceptions since they can be made metallic. Static stripes seem exceptional in cuprates: they occur at the crystal surfaces (the scanning tunneling microscopy measurements) while only in very special (214-LTT) cuprates they seem to exist in the bulk, where they are very bad for the superconductivity. So much is clear that in the cuprates stripes are shaken hard by quantum fluctuations. The density matrix renormalization group computations by Steve White and Doug Scalapino in the mid 1990's were quite revealing in this regard.⁴⁴ Different from the 'deeply classical' Hartree-Fock stripes, the stripes in the t-J model seem to be governed by a RVB like organization of the spin system, such that they can be viewed as in essence a crystallization of valence bond pairs.⁴⁵ Quantum physics also seems to rule the stripes in the long time limit. Resting on the information obtained on the dynamical spin fluctuations by inelastic neutron scattering the case has been forcefully made⁴⁶ that the quantum liquid in underdoped cuprates looks like a stripe state that has fallen prey to what the quantum field theorists call 'dynamical mass generation'. These dynamical stripes might behave as static stripes at short distances but going to longer times the *collective* quantum fluctuations associated with the order increase, to get out of hand at a relatively long time, beyond which one is dealing with a uniform quantum liquid.

It has proven difficult to get a handle on these dynamical stripes. However, inspired by this question a next great idea was born. Static stripes should in first instance be seen as an electron crystal. Resting on the analogy with the 'partial' melting of classical crystals due to thermal fluctuations, Steve Kivelson, Eduardo Fradkin and Vic Emery introduced in 1997 the idea⁴⁷ that, given the severe quantum fluctuations, the stripe crystal in the cuprates could perhaps quantum melt step by step. This would then yield a hierarchy of quantum liquid crystal states: the quantum smectic (crystal in one direction, quantum liquid in the others) and the quantum nematic (a quantum liquid breaking spontaneously rotational symmetry).

I was immediately enchanted by the idea. Realizing that this was potentially a highly fertile territory, I made it into a main focus of my own research, but as far as I am aware this was for a long time the only substantive effort outside the Kivelson-Fradkin group. The lack of interest in this obviously interesting pursuit was quite frustrating but the show got eventually saved again by the experimentalists. Right now the quantum liquid crystals are turning into a fashionable research area. In the cuprates the first evidences were produced by the inelastic neutron scattering work by the group of Keimer, followed by the very recent work of Tranquada *et al.* on the 'sliding phase' physics in the LTT cuprates, and especially the Nernst effect anisotropy discovered by Taillefer's

⁴⁴S.R. White and D.J. Scalapino, Phys. Rev. Lett. 80 (1998) 1272.

⁴⁵As first suggested by S. Sachdev and N. Read, Int. J. Mod. Phys B 5 (1991) 219: this appears to be quite consistent with the $Cu-O-Cu$ 'valence bond' units comprising the stripes seen by the STS 'Mott maps' by the Davis group.

⁴⁶Our Leiden group was in this regard (too) far ahead of the time, investing much energy in the problem of fluctuating stripes in the early 1990's: See J. Zaanen, M.L. Horbach and W. van Saarloos, Phys. Rev. B 53 (1996) 8671 (1996) and the review J. Zaanen *et al.* Phil. Mag. B 81 (2001)1485. See also S.A. Kivelson *et al.*, Rev. Mod. Phys. 75 (2003) 1201 and for a recent review see: M. Vojta, Adv. Phys. 58 (2009) 699.

⁴⁷S.A. Kivelson, E. Fradkin, and V.J. Emery, Nature 393 (1997) 550.

group and the peculiar 'hidden' orientational symmetry breaking in the STM stripes of Davis et al. This is supplemented by the particularly neat quantum nematic found at the metamagnetic quantum critical point in a ruthenate by the group of Mackenzie, as well as the very recent evidence for the famous hidden order in URu_2Si_2 to be of this kind. Perhaps the most spectacular evidence was presented in 2010, in the form of the extreme transport anisotropies in detwinned underdoped pnictides by the Fisher group.

I find the greatest star among the exotic pseudogap orders to be the spontaneous diamagnetic current phase that started with the theoretical work by Chandra Varma.⁴⁸ This opinion is motivated by the fact that this type of order is completely detached from anything that was known before 1987. The essence of the phenomenon is that below the pseudogap temperature diamagnetic currents start to flow spontaneously around plaquettes in the copper-oxide lattice.

The idea that something of this kind can happen dates back to the wild early years of high- T_c superconductivity, where it arose in the context of the gauge theories discussed in section 2.4.4. In their study of the slave mean field theories controlled by gauge symmetry, Ian Affleck and Brad Marston⁴⁹ observed that besides the spinon d-wave superconductor there was another stable mean field state. In this 'staggered flux phase' diamagnetic currents occur spontaneously, forming a staggered pattern on the plaquettes of the square lattice. In the context of the gauge theories this flux phase has a very special status: in Wen's modern projected symmetry language, the d-wave superconductor and the flux phase span up the gauge volume in the sense that one can *locally* rotate one in the other without physical ramifications for the spin liquid. Upon doping this gauge invariance is lifted and the flux phase turns into a real phase that takes the role of competitor of the d-wave superconductor. A main result of the work by Patrick Lee and coworkers on the $SU(2)$ formulation is the prediction that one should find collective modes in the d-wave superconductor that correspond with exciting infinitesimal flux phase modes.⁵⁰

This idea was recycled in 1999 by Chetan Nayak⁵¹ showing that staggered flux phases have a generic significance also in a more conventional (Hartree-Fock) setting. It is well known that under certain fine tuning conditions one can store the particle-particle Cooper pairs and particle-hole charge density wave in a single (global) $SU(2)$ multiplet (Anderson's pseudo-spins), where the superconductivity is associated with the XY plane and the CDW with the z-direction of the pseudo-spin internal space. In this way one can show that the $SU(2)$ partner of a s-wave superconductor is a conventional charge density wave. Nayak observed that a d-wave superconductor 'rotates' in a similar guise into a staggered flux phase. This motivated the name d-density wave (d-DW), and it gave further impetus to the idea that as such it would be a natural candidate for the 'hidden' competing pseudo-gap order. However, despite a concerted experimental effort this idea failed to generate any empirical support.

Varma actually departed from a quite different perspective. From the very beginning of the cuprate odyssey he had been in the grip of the idea that one has to look in more detail into the chemistry of the cuprate planes, taking a quite dissident stance in a climate dominated by the divinity of the simple t-J model. Varma departed from the wisdom that the holes are physically located on the oxygen ions, while he observed that in the presence of 'excitonic' Cu-O nearest neighbor Coulomb interactions one can run into instabilities when the doping increases. Much later, he realized that such microscopic physics opens up the possibility for d-DW type currents to develop inside the Cu-O unit cell, forming patterns involving the various $O - Cu - O$ intra-unit cell triangular plaquettes. The specialty of such a phase is that it breaks only time reversal symmetry, with the ramification that one has to know where to look for it in experiment in order to find it. The first indications for

⁴⁸C.M. Varma, Phys. Rev. Lett. 83 (1999) 3538.

⁴⁹I. Affleck and J.B. Marston, Phys. Rev. B37 (1988) 3774.

⁵⁰P.A. Lee and N. Nagaosa, Phys. Rev. B68 (2003) 024516.

⁵¹C. Nayak, Phys. Rev. B62 (2000) 4880; this was quite effectively advertised by S. Chakravarty *et al.*, Phys. Rev. B63 (2001) 094503.

the presence of such a phase in the pseudogap regime were derived from ARPES measurements, but the data analysis turned out to be a contentious affair. This landed on its feet in 2005 when Philippe Bourges *et al.* found direct evidence for such intra-unit cell 'antiferromagnetism' breaking time reversal symmetry using rather straightforward spin-polarized neutron scattering. Around the same time complimentary evidences were found by Kapitulnik *et al.* employing highly sensitive optical dichroism measurements.

I was among the first to see this paper – as 'Mr. Stripes' I has perhaps something to loose, but I was immediately convinced that Varma's currents are real. Given the persuasiveness of the evidence, and the fact that this was a genuine new type of ordering phenomenon I took it for granted that this would make headlines by the next day. But reality was quite different. Again the brick wall of hard headed conservatism was erected. The authors had a hard time to get this published and initially it was largely ignored. As with the stripes and the nematics, the neutron scattering community once again demonstrated its open mindedness. In a concerted effort, with a particularly prominent role of Martin Greven with his superb mercury crystals, much additional evidence was generated. The latest evidence is by Philippe Bourges *et al.*, showing that in the 214 stripe phase the stripes not only live together with nematic order (according to Davis *et al.*), but also with a short ranged Varma current order, making the case that the pseudogap phase deserves a name that was recently conceived by Uchida: the monster.

Summarizing, although the pseudogap regime has still its mysterious sides (like the normal state 'Fermi-arcs', and the 'two gap' behavior) the theorists can at the least claim that they have played a decisive role in uncovering the various exotic ordering phenomena that are taking place. Another matter is what this history reveals regarding the sociology of this community. As I implicitly emphasized, to my perception the community operates in this regard in a less than optimal way. Scientific skepticism is a great good and perhaps it is just part of human nature that scientific communities have to be conservative. However, an atmosphere that would have been a bit more receptive towards exciting new scientific potentialities would have undoubtedly rendered the above research enterprise to be a more efficient affair.

1.6 Quantum critical metals and superconductivity.

A last major theme in the context of theories of unconventional superconductivity is the school of thought that revolves around the idea that the physics of the cuprates, heavy fermion superconductors, and perhaps the pnictides and organics revolves around a quantum phase transition. The emphasis is on 'quantum': the transition is happening at zero temperature and driven by quantum fluctuations. In a way this general idea has many fathers. It is also a prevalent theme elsewhere in physics: I recently learned that in string theory the notion of what is known in condensed matter as quantum criticality is among the fundamental organizing principles in this seemingly very different area of physics. In fact, I will end this history reporting on a very recent attempt to mobilize the powerful mathematics of string theory to corner the physics of high- T_c superconductivity.

Symmetry is the best friend of the theoretical physicists and the critical state (quantum or not) is in the grip of one of the most powerful symmetries that has been identified: scale invariance. This came to flourish in the classical realms in the 1970-1980's by the discovery of Wilson's renormalization group theory, which is in essence just the theory that exploits the power of the scale invariance emerging at the classical continuous phase transitions. In a way it is remarkable that it took so long before it was realized how to exploit its full powers in the quantum realms. In high energy physics this is likely due to the pre-occupation with QCD. This is about the 'marginal' case with its running coupling constants, as echoed in condensed matter physics in the form of the attention to the

Kondo impurity problem and the non-abelian 'Haldane' spin chains subjected to the phenomenon of dynamical mass generation. It appears that in string theory conformal field theory (also beyond the 1+1D worldsheet context) came fully into focus when supersymmetric field theories were explored. Due to the non-renormalization theorems coming with supersymmetry these are characterized by strongly interacting unstable fixed points (in the statistical physics jargon) that span up planes in the space of coupling constants.

In condensed matter physics the idea of quantum phase transitions entered the main stream in 1988, by the seminal work of Sudip Chakravarty, David Nelson and Bert Halperin (CNH).⁵² Bert Halperin told me that this work was initially motivated by his conviction that Phil Anderson got it all wrong with his claim that the magnetism in the Mott insulating La_2CuO_4 had to do with resonating valence bonds. Bert got it right: the quantum magnetism in this system is of the quasiclassical kind, controlled by the collective quantum precessions of the Néel order parameter. Compared to many other antiferromagnets, this $S = 1/2$ two dimensional Heisenberg spin system is not that far from the point where these precessions get completely out of hand, quantum melting the Néel order in a continuous quantum phase transition. This physics is captured by the $O(3)$ quantum non-linear sigma model field theory which amounts to a 'classical' Heisenberg system that is living in the 2+1 dimensional space time of the euclidean (thermal) path integral. Sudip Chakravarty was back then on sabbatical in Harvard and it appears that due to his high energy physics training he was pivotal in the realization that this becomes really interesting when one asks what the proximity to the quantum phase transition means for the *finite temperature* physics. As far as I am aware they were the first to realize that by raising temperature one re-enters the quantum critical 'wedge' where the low energy physics is controlled by the zero temperature quantum physical scale invariance. They demonstrated how the spin stiffness controlling the thermal order parameter fluctuations is renormalized by the quantum fluctuations as was soon thereafter spectacularly confirmed experimentally.

In the same year the Bell Labs theorists Varma and Littlewood with coworkers had independently constructed the marginal Fermi liquid phenomenology for the metallic states of cuprates at optimal doping.⁵³ This is actually not much of a mathematical theory, but it was quite effective in highlighting- and unifying the strange behaviors of these normal state metals in a single phenomenological (or 'heuristic', 'mnemonic') framework. Forced by the experimental circumstances Varma *et al.* imported the 'Planckian dissipation' (energy/temperature scaling) as a crucial building block of the construction, among others giving a heuristic explanation for the famous linear-in-temperature resistivity of the optimally doped metals.

Subir Sachdev discovered soon afterward that the classical, relaxation hydrodynamics inside the quantum critical wedge is highly unusual⁵⁴. Much later I named it 'Planckian dissipation'⁵⁵: it is a dissipative state, controlled by Planck's constant through a 'universal' relaxation time $\tau_h \simeq \hbar/(k_B T)$. In my own reference frame I became fully aware of the meaning of quantum criticality-proper at

⁵²S. Chakravarty, D.R. Nelson and B.I. Halperin, Phys. Rev. B39 (1989) 2344; In this same era the notion of 'quantum critical hydrodynamics' was also fully realized in the context of DC transport phenomena both involving (fractional) quantum Hall plateau transitions (S.L. Sondhi *et al.*, Rev. Mod. Phys. 69 (1987) 315) and the superconductor-insulator transition in amorphous systems (M.P.A. Fisher *et al.*, Phys. Rev. B40 (1989) 546; *ibid.*, Phys. Rev. Lett. 64 (1990) 587).

⁵³C. M. Varma *et al.*, Phys. Rev. Lett. 63 (1989) 1996. Undoubtedly Phil Anderson also played a key role in attracting attention to the strangeness of the normal state metal. With his 'tomographic Luttinger liquid' (see his book) and his more recent 'hidden Gutzwiller quasiparticle gas' (Nature Physics 4 (2008) 208) he was (and is) searching for a truly non perturbative formulation for a state that behaves temporally in a critical way while it is still 'organized around a Fermi surface'. This contrasts strongly with the perturbative (Hertz-Millis like) attitude of the marginal Fermi-liquid. At least on the phenomenological level his constructions are remarkably similar to the very recent results coming from the Anti-de-Sitter/Conformal Field Theory correspondence. Historically the marginal Fermi-liquid had perhaps more impact for no other reason than its lack of pretense, to be no more than a way to memorize the strangeness in the experimental data.

⁵⁴S. Sachdev and J. Ye, Phys. Rev. Lett. 69 (1992) 2411

⁵⁵J. Zaanen, Nature 430 (2004) 513; embarrassingly, I called such a state maximally viscous not realizing that viscosity is proportional to the relaxation time. The Planckian dissipator is therefore as close to a perfect (un-viscous) fluid as possible, as was realized in the same period in the context of string theory and the quark-gluon plasma.

the Aspen winter conference in 1992 where Sachdev presented their analysis of the NMR spin-spin and spin-lattice relaxation times in superconducting, underdoped cuprates employing the Planckian dissipation.⁵⁶ The implication of this work was that at least the spin fluctuations of the optimally doped 'marginal Fermi-liquid' was in the grip of the quantum scale invariance. This was an important moment in my career: I have been since then a devotee of the quantum critical state, in fact as a goal by itself regardless its relevance towards superconductivity and so forth. During the 1990's Sachdev developed the quantum critical hydrodynamics theme further and this culminated in his quantum phase transition book.⁵⁷

The theme was picked up in the high- T_c community, since the relevancy of the notion towards the then widely accepted marginal Fermi-liquid phenomenology was rather obvious. A difficulty was however that for quantum criticality to control the physics at optimal doping it occurs that one needs a quantum phase transition, and for this one needs some form of order that comes to an end in the best superconductors. Loram and Tallon came up early on with thermodynamical evidences for such a quantum critical point but this was back then highly controversial. As described in the previous section in the last few years this has changed drastically with the solid empirical evidences for the 'monster' competing orders that are simultaneously at work in the underdoped regime. It remains to be seen whether a single order parameter is responsible for the quantum critical regime in the sense of a 'conventional' isolated quantum unstable fixed point. A very recent idea is that the competing orders of the underdoped regime are more an effect than a cause, reflecting instead the collapse of the Mott-ness at optimal doping which is the real driving force behind the quantum criticality at optimal doping.⁵⁸ Altogether, the idea that the cuprate superconductivity is rooted in the quantum critical nature of the optimally doped metal is at present again very popular but more work needs to be done to nail down how it precisely works. The empirical situation is in this regard more transparent in the development that came next: the heavy fermions.

This unfolded in the early 1990's with the discovery of a plethora of quantum phase transitions in the heavy fermion intermetallics, as triggered by the seminal work of the Lonzarich group at Cambridge.⁵⁹ These quantum phase transitions involve rather straightforward magnetic order that is disappearing. However, on both sides of such quantum critical points (QPT) one finds Fermi-liquids that are characterized by a quasiparticle mass that tends to diverge at the critical point. The fermions are in one or the other way 'on the critical surface' and perhaps the most dramatic empirical fact is that one invariably finds that a 'last minute' instability sets in, centered at and 'shielding' the quantum critical point. Usually this corresponds with an unconventional superconductor that takes over from a normal state showing the traits of a quantum critical metal like a linear resistivity. Since these systems involve the localized f electrons of lanthanides or actinides the microscopic scales are much smaller than in the cuprates. All along it has been realized that by 'boosting' up their microscopic energy cut-off, the physics of the heavy fermion systems would be strikingly similar to that of the cuprates, adding credibility to the idea that also in the latter case the superconductivity is eventually rooted in the quantum criticality itself.

As discussed in section 2.4.2, the idea that spin fluctuations are responsible for the cuprate superconductivity has been around since the very beginning. Catalyzed by the discoveries in the heavy fermion systems this merged with the quantum criticality idea, actually falling back on a theoret-

⁵⁶A.V. Chubukov and S. Sachdev, Phys. Rev. Lett. 71 (1993) 169.

⁵⁷S. Sachdev, *Quantum Phase Transitions*, Cambridge Univ. Press (1999).

⁵⁸As argued by for instance Philip Phillips, the Hubbard projections might come to an end for increasing doping when U is finite (Rev. Mod. Phys. 82 (2010) 1607; for closely related ideas in the heavy fermion context see C. Pepin, Phys. Rev. Lett. 98 (2007) 206401). As we argued (arXiv:0911.4070), the big Fermi-surface in the overdoped regime might be taken as evidence that the Hubbard projections have disappeared in the overdoped regime (in the spirit of the considerations in Anderson's book) while the Mott-ness dominated underdoped regime falls prey to competing orders because of the altered 'Weng quantum statistics'. The ramification would be that the quantum criticality is rooted in a quantum-statistics clash between the different statistics of the Mott-ness and Fermi-liquid regimes.

⁵⁹see, e.g., J. Zaanen, Science 319 (2008) 1205 and references therein.

ical interpretation of quantum criticality that was introduced many years before CNH entered the stage. This dates back to the development of paramagnon theory discussed in the second section. Surely, in the context of itinerant magnetism paramagnons become significant when one gets close to the transition to ordered magnetism. In 1976 John Hertz reformulated this in the modern thermal field theory language, respecting the essence of quantum criticality, but eventually ending up with a story that is operationally similar to the RPA treatment (the Moriya paramagnon theory).⁶⁰ It starts out with the weakly interacting Fermi-gas, the Hubbard-Stratonovich order parameter fields are introduced, and it is asserted that the fermions can be integrated out on the RPA level, forming a heat bath damping the order parameter. This lifts the effective dimensionality of Euclidean space time felt by the order parameter dynamics such that it lands at- or above its upper critical dimension, implying that the fixed point is Gaussian. In the modern era it was realized that these order parameter fluctuations 'backreact' on the fermions in the form of causing a singular interaction in the pairing channel.⁶¹ This in effect boils down to a fancy version of the spin fluctuation 'superglue' idea, where now the spin fluctuations appear in a quantum critical incarnation causing the superconducting dome surrounding the quantum critical point.

This has turned into a quite popular view both in the heavy fermion and cuprate communities. It can even be correct but the problem is that the construction rests on a manifestly uncontrolled approximation scheme. The fundamental problem are the fermion signs. Regardless the representation, fermions are not governed by an effective Boltzmannian physics in Euclidean space time due to the destructive workings of the Fermi-Dirac statistics rendering the quantum partition sum to be one with mathematically ill behaved 'negative probabilities'. In dealing with a fermionic critical state one can therefore a-priori not fall back on the 'hidden' classical criticality, which is the secret behind the rigor of the CNH/Sachdev treatment of the bosonic problem. In the Hertz treatment this 'fermion nightmare' is worked under the rug in the beginning, by the assumption that the fermions can just be integrated out. The flaw is that the integration of the fermions requires them to be massive but the fermion sector is massless and therefore the Fermi surface itself is somehow part of the zero temperature critical state. In fact, it has become clear more recently that in some heavy fermion systems the Fermi surfaces reconstruct drastically at the quantum critical point, demonstrating directly that the fermionic degrees of freedom are "active" part of the criticality.⁶²

With regard to the key question of why T_c is as high as it is we have arrived at the present day. Although one still finds plenty of strong opinions regarding the truth of spin fluctuations or the divinity of RVB in the community, the 'agnostic view' I referred to in the beginning has become steadily more influential in the community. At least my own perspective has been changing considerably in the last quarter century, from the desire to just explain why T_c is high to the conviction that the cuprates just have been instrumental in making visible a very deep and general problem in fundamental physics at large. The modern code word is 'quantum matter', the question regarding the general nature of matter as governed by the fundamental laws of quantum physics and the general principles of emergence. Remarkably, it appears that physics is entering an era where the empirical condensed matter- and the mathematical high energy communities are merging their activities for a head on attack on this, in a way, very new problem.

Some twenty years ago there was an implicit but widespread believe that we knew what matter is. However, high- T_c superconductivity and related phenomena have played the role of the Michelson-Morley experiment, opening our eyes for the fact that we actually have no clue and that there is plenty of room for surprises. I already alluded a number of times to the 'fermion signs', the 'negative probabilities' that spoil the mathematics that we can handle, which also show up in various guises in problems that are not governed by fermion statistics (like frustrated spins, bosons when

⁶⁰J. Hertz, Phys. Rev. B14 (1976) 1165, with some corrections by A. Millis, Phys. Rev. B48 (1993) 7183.

⁶¹For a review see Ar. Abanov, A.V. Chubukov and J. Schmalian, Adv. Phys. 52 (2003) 119.

⁶²See the recent review: H. von Löhneisen *et al.*, Rev. Mod. Phys. 79 (2007) 1015.

time reversal is broken, any non-equilibrium quantum-field problem). The bottom line is that these 'quantum signs' are ubiquitous and in their presence we have no mathematical tools for the description of matter that really works. The exception are the incompressible states (like the fractional quantum Hall states, topological insulators) where topology is helping us out, but for compressible states it is just guess work: we just know from experiment that Fermi-liquids and the 'Hartree Fock states' (including those of section 2.4.5) are part of nature's portfolio of forms of matter.

The problem of fermionic quantum criticality is rather interesting from this viewpoint. Scale invariance is a powerful symmetry and it should somehow render the general sign problem to become more easy. However, departing from the fifty year old established mathematical technology one runs immediately into a deep problem: to have perturbative control one needs the fact that the Fermi-liquid has a scale (the Fermi-energy). What to think of a fermionic matter without Fermi energy, and how to think about a Fermi-surface when there are no quasiparticles? In the Hertz approach and so forth these questions are worked in the rug since these constructions are departing implicitly from the stability of the Fermi-liquid. We need obviously a mathematical weaponry to address these issues which is not lying on a shelf somewhere.

With the reservation that it is very fresh, it might be that in this regard 2010 will end in the history books as the year that it started to happen. During the last forty years, string theorists have been working hard developing some very new mathematical machines. Although their performance as mere mathematical constructs is breath taking, it has been quite frustrating that these string-machines do not seem to have an obvious application to explain observable phenomena in nature. But this is now changing: it appears that the most powerful machine of string theory (the Anti-de-Sitter/Conformal Field Theory correspondence) is just tailored to compute the properties of states of sign-full quantum matter. The way this works is at first sight outrageous, but it becomes very beautiful when one get used to the idea. It departs from the notion that in a special sense black holes behave as material objects when combined with quantum physics. They carry entropy and so forth, and in their simplest incarnation they just behave as black body radiators giving off thermalized Gaussian fields. The modern string theoretical version has turned this into a 'generalized particle wave' duality, insisting in a surprisingly precise mathematical way that such black hole worlds in $d+1$ dimensional space times are in a dual sense equivalent to strongly interacting forms of quantum matter living in a boring flat d -dimensional space time – the stuffs that one finds in the earthly material science and cold atom labs!

Right now the string theorists are sorting out collections of colorful 'hairy black holes' that by themselves represent a leap forward in quantum gravity. These in turn render a description of quantum matter that extends way beyond the fifties paradigm. It started with the recognition in 2007 by Sachdev, Son and coworkers⁶³ that the correspondence encodes the 'Planckian dissipation' hydrodynamics associated with the finite temperature quantum critical state in the condensed matter systems. Since last year matters this development speeded up, by the discovery that Fermi-liquids emerge 'out of the blue' from a strongly interacting critical ultraviolet, while their stability is encoded by a quantum-mechanical 'Dirac hair' hanging some distance way from the black hole horizon.⁶⁴ However, such 'Dirac hair' can be 'shaved away' and the resulting 'extremal' Reissner-Nordstrom black-hole codes for an emergent quantum critical fermion state. This is breathtaking: it describes an infrared that has a sole temporal scale invariance, while the 'matching' with the bare ultraviolet fermion generates a singularity structure in momentum space that corresponds with the generalization of the Fermi-surface to the non-Fermi-liquid realms⁶⁵ This is quite like the marginal Fermi-liquid, which was introduced on mere empirical grounds!

⁶³C.P. Herzog *et al.*, Phys. Rev. D75 (2007) 085020, see also J. Zaanen, Nature 448 (2007) 1000 for a light perspective.

⁶⁴M. Cubrovic, J. Zaanen and K. Schalm, Science 325 (2009) 439.

⁶⁵H. Liu, J. McGreevy and D. Vegh, arXiv:0903.2477, T. Faulkner *et al.*, Science 329 (2010) 1043; see also S. Sachdev, arXiv:1006.3794 for inspirational ideas of how this relates to the matters discussed in section 2.4.4.

Last but not least, the 'hairy black hole' pursuit started in 2008 by the ' H^3 ' discovery⁶⁶ of the holographic encoding of the superconductor in terms of 'scalar hair'. This tells us a fascinating story of a 'glueless' BCS like superconductivity, where the superconducting instability is driven by a kind of 'perfect quantum frustration' of the non-superconducting quantum critical state (of the 'bald' extremal kind), as manifested by the fact that the 'naked' quantum critical point would carry a zero temperature entropy that is 'eaten' by the superconductor. This resonates with the idea that has been around in especially the heavy fermion community for quite some time: were it not for the intervention of the 'last minute' superconductivity one would have to deal with a Fermi-liquid acquiring an infinite quasiparticle mass right at the critical point, representing of course such a quantum frustration. The end of the present history is that right now the first attempts are undertaken to nail down this quantum frustration in the experimental laboratories.⁶⁷

I might be overly optimistic but I have to admit that the developments discussed in the last paragraphs have been a source of inspiration to do my best to formulate the present 'chronicles of the dark ages of quantum matter'. In case that the black hole dreams land on the real axis, the professional historians will be quite curious regarding the hairy details of the prehistory of this scientific revolution. When this fails, it doesn't matter since at some point in the future likely a very young physicist will scribble equations on a black board of a quality that in no time a general consensus will emerge that the mystery of high- T_c superconductivity is solved. Given our present understanding of the depth of the mystery there is no doubt in my mind that the insights gained from these equations should be at least as good as the black holes!

⁶⁶S.A. Hartnoll, C. Herzog and G. Horowitz, Phys. Rev. Lett. 101 (2008) 031601.

⁶⁷see J. Zaanen, Nature 462 (2009) 15.