Topics in Standard Model

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Electron scattering in Coulomb field (reminder from the previous lecture)

- In non-relativistic quantum mechanics if Hamiltonian has the form $\hat{H} = \hat{H}_0 + \hat{V}$ then the probability of transition between an initial state $\psi_i(x)$ and the final state $\psi_f(x)$ of **unperturbed** Hamiltonian $\hat{H}_0$ is given by (Landau & Lifshitz, vol. 3, § 43):

$$dw_{if} = \frac{2\pi}{\hbar} |V_{if}|^2 \delta(E_i - E_f)dn_f$$  \hspace{1cm} (1)

where $|V_{if}|$ is the matrix element between initial and final states; and $dn_f$ is the number of final states with the energy $E_f$ (degeneracy of the energy level).

- In the case of Dirac equation, the interaction is given by

$$V_{\text{int}} = \int d^4x \bar{\psi}(x) \gamma^\mu A_\mu(x) \psi(x)$$  \hspace{1cm} (2)

recall that electric current $j^\mu = \bar{\psi}(x) \gamma^\mu \psi(x)$

— Following Bjorken & Drell, Sec. 7.1

Alexey Boyarsky  \hspace{1cm} STRUCTURE OF THE STANDARD MODEL
Electron scattering in Coulomb field (reminder from the previous lecture)

If we consider static point source with the Coulomb field

\[ A_0(x) = \frac{Ze}{4\pi|x|} \quad (3) \]

and wave-functions\(^1\)

\[ \psi_i(x) = \sqrt{\frac{m}{E_i V}} u_s(p_i)e^{-ip_i \cdot x}, \quad \bar{\psi}_f(x) = \sqrt{\frac{m}{E_f V}} \bar{u}_r(p_f)e^{ip_f \cdot x} \quad (4) \]

\( (E_i = E_f) \)

Following (1) we write the matrix element

\[ V_{if} = \frac{Ze^2}{4\pi} \frac{1}{V} \sqrt{\frac{m^2}{E_i E_f}} \bar{u}_r(p_f)\gamma^0 u_s(p_f) \int d^3 x e^{ix \cdot (p_i - p_f)} A_0(x) \quad (5) \]

\(^{1}\)Here \(u_s, \bar{u}_r\) are 4-component spinors – solution of the Dirac equations \((\gamma \cdot p - m)u_s = 0, \bar{u}_r(\gamma \cdot p + m) = 0, s = \pm, r = \pm\) – polarizations of spin.
**Electron scattering in Coulomb field (reminder from the previous lecture)**

- Degeneracy of a final state with $E_f$ is given by

\[
dn_f = 2 \times \int_{p_0 > 0} d^4 p \delta(p^2 - m^2) = \frac{d^3 p_f}{(2\pi)^3 E_f} \tag{6}
\]

- As a result we get

\[
dw_{if} = 2\pi |V_{if}|^2 \frac{d^3 p_f}{(2\pi)^3 E_f} \tag{7}
\]

\[
= \frac{Z^2 (4\pi \alpha)^2 m^2 |\bar{u}_r(p_f)\gamma^0 u_s(p_f)|^2}{E_i V} \frac{d^3 p_f}{(2\pi)^3 E_f} \delta(E_i - E_f)
\]
Consider next the situation when the electromagnetic field is created by other particle (“proton”)

While the formulas (4)–(6) remain true, the expression for $A_\mu$ changes.

If proton is described by a spinor $\Psi$, then its electric current is

$$J^\mu(y) = \bar{\Psi}(y) \gamma^\mu \Psi(y)$$

(8)

(the form of $\Psi_i$ and $\bar{\Psi}_f$ is the same as Eq. (4) with $m \rightarrow M_p$ and different momenta)

The electromagnetic field obeys the Klein-Gordon equation

$$\Box A_\mu = J_\mu \quad \text{or} \quad A_\mu(x) = \frac{1}{\Box} J_\mu(y)$$

(9)
Electron scattering on proton

Very naively, the operator $\Box^{-1}$ (inverse to the Klein-Gordon operator) can be easily constructed if one considers (9) in Fourier space:\footnote{We denote by $\tilde{A}_\mu(p)$ and $\tilde{J}_\mu(p)$ Fourier transform}

\[
p^2 \tilde{A}_\mu(p) = \tilde{J}_\mu(p) \quad \text{or} \quad \tilde{A}_\mu(p) = \frac{1}{p^2} \tilde{J}_\mu(p)
\]

Therefore, the solution of Eq. (9) with arbitrary source term is

\[
A_\mu(x) = \int \frac{d^4 p}{(2\pi)^4} e^{ip\cdot x} \frac{\tilde{J}_\mu(p)}{p^2} = \int \frac{d^4 p}{(2\pi)^4} \int d^4 y \frac{e^{ip\cdot (x-y)}}{p^2} J_\mu(y)
\]

Notice that in Eq. (5) we only need $\tilde{A}_\mu(p_f - p_i)$. The resulting expression is then equivalent to (7) if one substitutes

\[
\gamma^0 \frac{Z}{|q|^2} \to \gamma^\mu \frac{1}{q^2} \sqrt{\frac{M_p^2}{E_i(p) E_f(p)}} \bar{U}_r(P_i) \gamma_\mu U_s(P_f)
\]

where $q = p_f - p_i = P_i - P_f$ – transferred 4-momentum and $q$ is its spatial component. 4-spinors $U_s$ and $\bar{U}_r$ are the in- and out- 4-spinors of a proton.
Electron scattering on proton

That is the result looks like a scattering of electron in external field (7) where the external field $A_\mu$ is the field created by the proton (11) and (8).

So far we have considered two processes

- Electron scatters on static electric field
- Electron scatters on dynamic electromagnetic field, created by another moving particle (proton).

What changes if instead of electromagnetic field we are taking real photons?
Let us now come back to Eq. (10) where we computed an “inverse of Klein-Gordon operator”. This object is known as Green’s function.

Have we encountered Green’s function before?

Yes! In the non-relativistic Quantum Mechanics one considers solving the Schrödinger equation in perturbation theory. Namely, given a free Hamiltonian $\hat{H}_0$ with the set of the eigen-function $\psi_n^{(0)}$ and perturbation $\hat{V}$ one can construct perturbed wave-functions (see e.g. Landau & Lifshitz, vol. 3, § 38)

$$\psi_n^{(1)} = - \sum_{k \neq n} \frac{V_{kn}\psi_k^{(0)}}{E_k - E_n}$$

To see that this is the same as Eq. (30), let us rewrite the expression
Green’s function of the Schrödinger equation

\((\hat{H}_0 + \hat{V})\psi = E\psi\) similarly to (28):

\[
\psi = -\frac{1}{\hat{H}_0 - E}\hat{V}\psi
\]  

(13)

The object

\[
\hat{G}_E \equiv \int dn \frac{|n\rangle \langle n|}{E_n - E}
\]  

(14)

(where \(|n\rangle\) is the full set of eigen-functions of \(\hat{H}_0\) ) has a special meaning.

Namely, consider a Hamiltonian \(\hat{H}\). Then the following expression holds:

\[
(\hat{H} - E)\hat{G}_E = \int dn \ |n\rangle \langle n| = 1
\]  

(15)

i.e. this is Green’s function of the operator \(\hat{H}\)

In \(|x\rangle\) representation

\[
G_E(x, x') = \langle x | \hat{G}_E | x' \rangle = \int dn \frac{\psi_n(x)\psi^*_n(x')}{E_n - E}
\]  

(16)
Green’s function of the Schrödinger equation

\[(\hat{\mathcal{H}} - E)G_E(x, x') = \delta(x - x')\]  \hspace{1cm} (17)

**Demonstrate** that the Green’s function of the free Schrödinger equation is

\[G_E(x, x') = \int \frac{d^3p}{(2\pi)^3} \frac{e^{ip \cdot (x-x')}}{2m - E} \]  \hspace{1cm} (18)

Obtain it from definition (14) and (18) using \(|n\rangle = |p\rangle\) and taking into account that \(\langle x | p \rangle = \frac{1}{\sqrt{V}} e^{ip \cdot x}\).
Similarly, the Green’s function of the Dirac equation is defined in the Fourier space as

\[
\frac{1}{\hat{D}^{(0)}} \equiv \tilde{S}_F(p) = \frac{1}{\gamma \cdot p - m}
\]  \hspace{1cm} (19)

acting on an arbitrary function \( \psi(x) \) as

\[
\langle x | \frac{1}{\hat{D}^{(0)}} | \psi \rangle = \int \frac{d^4p}{(2\pi)^4} \int d^4x' \frac{e^{-ip \cdot (x-x')}}{\gamma \cdot p - m} \psi(x')
\]  \hspace{1cm} (20)

Notice, that unlike the previous (scalar) Green’s function, this time we have a 4 \( \times \) 4 matrix in spinor indices in the denominator of (19).

\(^3\)Following Bjorken&Drell, we will use the notation \( S_F \) for the Green’s function of the Dirac equation.
Green’s function of the Dirac equation

The expression $\frac{1}{\gamma \cdot p - m}$ with matrices in the denominator should be understood in the following sense:

$$\frac{1}{\gamma \cdot p - m} = \frac{(\gamma \cdot p + m)}{(\gamma \cdot p - m)(\gamma \cdot p + m)} = \frac{(\gamma \cdot p + m)}{p^2 - m^2}$$  \hspace{1cm} (21)$$

Show that the r.h.s. of (21) is just an inverse matrix of the $4 \times 4$ matrix $(\gamma \cdot p - m \mathbb{I})$.

Green’s function in real space:

$$S_F(x - x') = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip \cdot (x-x')}}{p^2 - m^2}(\gamma \cdot p + m)$$  \hspace{1cm} (22)$$
Retarded/Advanced Green’s functions

- When computing the Fourier transform of $\frac{1}{p_0^2 - p^2 - m^2}$ we encounter a pole for all $p_0 = \pm \sqrt{p^2 + m^2}$. (the same problem we actually had with the Green’s function of the Schrödinger equation (18) — it has a pole on the real axis whenever $\frac{p^2}{2m} = E$)

- The poles of the Green’s function have deep physical meaning — they tell you what kind of particles exist in the theory.

- Recall, that the Green’s function defines a solution of differential operator $\hat{L}$ with a $\delta$-functional source. In our case

$$\hat{L}_x G(x, x') = \delta^{(4)}(x - x')$$  \hspace{1cm} (23)

- Based on boundary conditions (in time) one can define retardged
Retarded/Advanced Green’s functions

and advanced Green’s functions:

\[ G_{\text{Ret}}(x - x') = 0 \quad \text{if} \quad t < t' \]
\[ G_{\text{Adv}}(x - x') = 0 \quad \text{if} \quad t > t' \]  \hspace{1cm} (24)

Their meaning is clear: if at the moment \( t = 0 \) in the point \( x = 0 \) we turned on a perturbation ("hit a system with a hammer") then there is a wave propagating through the system for \( t > 0 \). On the other hand, for \( t < 0 \) there is nothing , and therefore \( G_{\text{Ret}}(x - x') = 0 \) for \( t < t' \). (the advanced Green’s function is simply anti-causal, demonstrating time-reversal symmetry)

To define a retarded Green’s function of the Schrödinger equation one adds a “prescription” of how to compute the integral (18):

\[
\text{Schrödinger} \quad G_{\text{Ret}}(x - x') = \int \frac{dE}{2\pi} e^{-iE(t-t')} \int \frac{d^3p}{(2\pi)^3} \frac{e^{ip \cdot (x-x')}}{p^2 / 2m - E - i\epsilon} \]  \hspace{1cm} (25)
The symbol $i\epsilon$ should be understood as follows

$$\lim_{\epsilon \to +0} \int_{-\infty}^{\infty} \frac{dE}{2\pi i} e^{-iE\tau} \frac{1}{E + i\epsilon} = \theta(\tau) \quad \text{(26)}$$

where $\theta(\tau)$ is step-function, the integral is performed in the complex plane $E = E' + iE''$ and contour is chosen depending on the sign of $\tau$ as shown in Fig.

Compute the Green’s function of non-relativistic particle in the $(t, x)$ space (i.e. compute the integrals in (25))
In case of Dirac equation we have particles and anti-particles.

Anti-particles (holes) are moving “backward in time” (when electron within the Dirac sea moves to the left, hole (empty space) moves to the right)

Recall:

Therefore, in case of the Dirac equation we expect to see retarded Green’s function for particles \((E > 0)\) and advanced Green’s functions for anti-particles \((E < 0)\)
Causal Green’s function

- This is realized by the following $i\epsilon$ prescription

$$S_F(x-x') = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot(x-x')}}{p^2 - m^2 + i\epsilon (\gamma\cdot p + m)}$$

(27)
Iterative solution of the Dirac equation

Consider the Dirac operator\( \hat{D}(0) \equiv \gamma \cdot \hat{p} - m \) to which one adds a small perturbation \( \hat{V} \) so that \( \hat{D} = \hat{D}(0) + \hat{V} \).

A formal solution of the full Dirac equation is given by the following expression:

\[
\psi = -\frac{1}{\hat{D}(0)} \hat{V} \psi
\]  

Eq. (28) is just a formal rearrangement of \((\hat{D}(0) + \hat{V}) \psi = 0\). However, written in this form, it allows for the solution by iterations.

Namely, a sequence of solutions \( \{\psi^{(n)}\}, n = 0, 1, \ldots \) where

\[
\psi^{(n)} = -\frac{1}{\hat{D}(0)} \hat{V} \psi^{(n-1)}
\]  

(and \( \psi^{(0)} \) is the solution of the unperturbed Dirac equation \( \hat{H}^{(0)} \)) will converge to the true solution of (28)
In practice one often restricts itself to first few iterations, assuming that the perturbation $\hat{V}$ is “small enough”.

**First order:**

\[
\psi^{(1)} = -\frac{1}{\hat{D}(0)}\hat{V}\psi^{(0)}
\]  

(30)

**Second order:**

\[
\psi^{(2)} = \frac{1}{\hat{D}(0)}\hat{V}\frac{1}{\hat{D}(0)}\hat{V}\psi^{(0)}
\]  

(31)

Recall that in non-relativistic Quantum Mechanics if the transition $i \rightarrow f$ is forbidden, the analog of Eq. (1) reads (c.f. Landau & Lifshitz, vol. 3, § 43):

\[
dw_{if} = \frac{2\pi}{\hbar} \left| \int dn \frac{V_{in}V_{nf}}{E_i - E_n} \right|^2 \delta(E_i - E_f)dn_f
\]  

(32)
Green’s function of the Dirac equation

where \( \int d\eta \) is the integral over any basis of intermediate states.

Consider now electron-positron scattering. Superficially it looks like electron-proton scattering we considered before (blue box).

However, there is additional contribution to \( e^+e^- \) scattering. Although \( e^- + e^+ \to \gamma \) is forbidden for real particles (\( \langle e^-, e^+ | \hat{V} | \gamma \rangle = 0 \)) if both energy and momentum are conserved.

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4See Bjorken & Drell, Sec. 7.9
Green's function of the Dirac equation

\[ \hat{V} = \int d^3x \left( \bar{\psi}(x) \gamma^\mu \psi(x) \right) \hat{A}_\mu(x) \]  
\hfill (33)

- the second order expression \textit{a la} Eq. (32) can be written:

\[
dw_{if} = \frac{2\pi}{\hbar} \left| \int \frac{d^3k}{(2\pi)^3} \frac{\langle \bar{\psi}_i \gamma^\mu \psi_i \rangle \langle 0 | \hat{V} | \gamma_k \rangle \langle \gamma_k | \hat{V} | 0 \rangle (\bar{\psi}_f \gamma^\mu \psi_f)}{E_i - \omega_k} \right|^2 \delta(E_i - E_f) dn_f
\]
\hfill (34)

the sum is over all photon intermediate states with arbitrary 3-momentum \( k \).

- Amplitudes of two processes should be \textit{added together} such that the probability of the process is proportional to \(|\mathcal{M}_1 + \mathcal{M}_2|^2\) rather than \(|\mathcal{M}_1|^2 + |\mathcal{M}_2|^2\) (interference terms are present).

\[
|\mathcal{M}|^2 = (\ldots) \quad \left[ i \frac{\bar{u}(p'_1)(-i\gamma^\mu)u(p_1)\bar{v}(q_1)(-i\gamma^\mu)v(q'_1)}{(p_1 - p'_1)^2} - i \frac{\bar{u}(p'_1)(-i\gamma^\mu)v(q'_1)\bar{v}(q_1)(-i\gamma^\mu)u(p_1)}{(p_1 + q_1)^2} \right]^2
\]
\hfill (35)

where \ldots is a prefactor, depending on energies/masses of particles.
Green’s function of the Dirac equation

— “Blue” diagram can be considered as a propagation of a particle (say, electron) in the electromagnetic field, created by the other particle (positron)
— “Red” diagram — electron-positron pair temporary disappears into a photon, photons “propagates” and the creates electron-positron pair

If we consider time as going upwards, then for the blue diagram at any given moment of time there exists electron + positron. For the “red” diagram, for some period of time only virtual photon exists