Topics in Standard Model

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Electron scattering in Coulomb field (reminder from the previous lecture)

- In non-relativistic quantum mechanics if Hamiltonian has the form $\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \hat{V}$ then the probability of transition between an initial state $\psi_i(x)$ and the final state $\psi_f(x)$ of **unperturbed** Hamiltonian $\hat{\mathcal{H}}_0$ is given by (Landau & Lifshitz, vol. 3, § 43):

$$dw_{if} = \frac{2\pi}{\hbar}|V_{if}|^2\delta(E_i - E_f)dn_f \tag{1}$$

where $|V_{if}|$ is the matrix element between initial and final states; and dn_f is the number of final states with the energy E_f (degeneracy of the energy level).

- In the case of Dirac equation, the interaction is given by

$$V_{\text{int}} = \int d^4x\,\bar{\psi}(x)\gamma^\mu A_\mu(x)\psi(x) \tag{2}$$

recall that electric current $j^\mu = \bar{\psi}(x)\gamma^\mu\psi(x)$

[0]Following Bjorken & Drell, Sec. 7.1

Electron scattering in Coulomb field (reminder from the previous lecture)

- If we consider static point source with the Coulomb field

$$A_0(\boldsymbol{x}) = \frac{Ze}{4\pi|\boldsymbol{x}|} \tag{3}$$

and wave-functions[1]

$$\psi_i(x) = \sqrt{\frac{m}{E_i V}} u_s(p_i) e^{-ip_i\cdot x} \quad , \quad \bar{\psi}_f(x) = \sqrt{\frac{m}{E_f V}} \bar{u}_r(p_f) e^{ip_f\cdot x} \tag{4}$$

($E_i = E_f$)

- Following (1) we write the matrix element

$$V_{if} = \frac{Ze^2}{4\pi}\frac{1}{V}\sqrt{\frac{m^2}{E_i E_f}} \bar{u}_r(p_f)\gamma^0 u_s(p_f) \int d^3\boldsymbol{x} e^{i\boldsymbol{x}\cdot(\boldsymbol{p}_i - \boldsymbol{p}_f)} A_0(\boldsymbol{x}) \tag{5}$$

[1] Here u_s, $\bar{u}_r$ are 4-component spinors – solution of the Dirac equations $(\gamma \cdot p - m)u_s = 0$, $\bar{u}_r(\gamma \cdot p + m) = 0$, $s = \pm$, $r = \pm$ – polarizations of spin.

Electron scattering in Coulomb field (reminder from the previous lecture)

- Degeneracy of a final state with E_f is given by

$$dn_f = \mathbf{2} \times \underbrace{\int_{p_0>0} d^4p\, \delta(p^2 - m^2)}_{\text{density of states}} = \frac{d^3\boldsymbol{p}_f}{(2\pi)^3 E_f} \tag{6}$$

- As a result we get

$$\begin{aligned} dw_{if} &= 2\pi |V_{if}|^2 \frac{d^3\boldsymbol{p}_f}{(2\pi)^3 E_f} \\ &= \frac{Z^2 (4\pi\alpha)^2 m^2}{E_i V} \frac{|\bar{u}_r(p_f)\gamma^0 u_s(p_f)|^2}{|\boldsymbol{p}_i - \boldsymbol{p}_f|^4} \frac{d^3\boldsymbol{p}_f}{(2\pi)^3 E_f} \delta(E_i - E_f) \end{aligned} \tag{7}$$

Electron scattering on proton

- Consider next the situation when the electromagnetic field is created by other particle ("proton")
- While the formulas (4)–(6) remain true, the expression for A_μ changes.
- If proton is described by a spinor Ψ, then its electric current is

$$J^\mu(y) = \bar{\Psi}(y)\gamma^\mu\Psi(y) \tag{8}$$

(the form of Ψ_i and $\bar{\Psi}_f$ is the same as Eq. (4) with $m \to M_p$ and different momenta)

- The electromagnetic field obeys the Klein-Gordon equation

$$\Box A_\mu = J_\mu \quad \text{or} \quad A_\mu(x) = \frac{1}{\Box} J_\mu(y) \tag{9}$$

Electron scattering on proton

- Very naively, the operator $\Box^{-1}$ (inverse to the Klein-Gordon operator) can be easily constructed if one considers (9) in Fourier space:[2]

$$p^2 \tilde{A}_\mu(p) = \tilde{J}_\mu(p) \quad \text{or} \quad \tilde{A}_\mu(p) = \frac{1}{p^2}\tilde{J}_\mu(p) \tag{10}$$

- Therefore, the solution of Eq. (9) with arbitrary source term is

$$A_\mu(x) = \int \frac{d^4p}{(2\pi)^4}\, e^{ip\cdot x}\frac{\tilde{J}_\mu(p)}{p^2} = \int \frac{d^4p}{(2\pi)^4}\int d^4y\, \frac{e^{ip\cdot(x-y)}}{p^2}J_\mu(y) \tag{11}$$

- Notice that in Eq. (5) we only need $\tilde{A}_\mu(p_i - p_f)$. The resulting expression is then equivalent to (7) if one substitutes

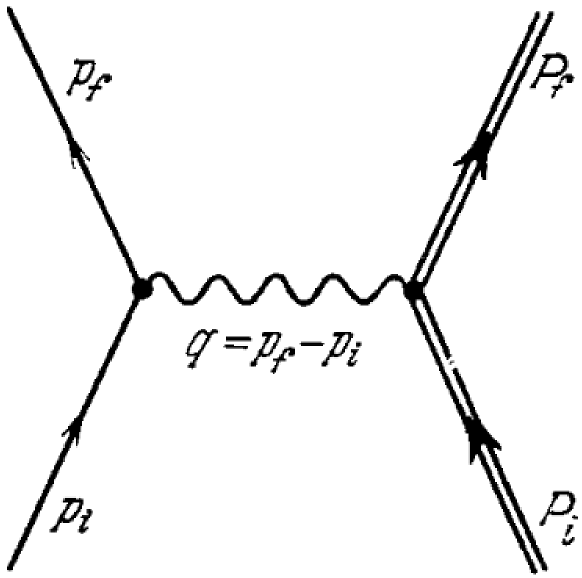
$$\gamma^0 \frac{Z}{|\boldsymbol{q}|^2} \to \gamma^\mu \frac{1}{q^2}\sqrt{\frac{M_p^2}{E_i^{(p)}E_f^{(p)}}}\bar{U}_r(P_i)\gamma_\mu U_s(P_f)$$

where $q = p_f - p_i = P_i - P_f$ – transferred 4-momentum and $\boldsymbol{q}$ is its spatial component. 4-spinors U_s and $\bar{U}_r$ are the in- and out- 4-spinors of a proton..

[2] We denote by $\tilde{A}_\mu(p)$ and $\tilde{J}_\mu(p)$ Fourier transform

Electron scattering on proton

- That is the result looks like a scattering of electron in external field (7) where the external field A_μ is the field created by the proton (11) and (8).



So far we have considered two processes

– Electron scatters on static electric field

– Electron scatters on dynamic electromagnetic field, created by another moving particle (proton).

- What changes if instead of electromagnetic field we are taking **real photons**?

Green's function of the Schrödinger equation

- Let us now come back to Eq. (10) where we computed an "inverse of Klein-Gordon operator". This object is known as **Green's function**

- Have we encountered Green's function before?

- **Yes!** In the non-relativistic Quantum Mechanics one considers solving the Schrödinger equation in perturbation theory. Namely, given a free Hamiltonian $\hat{\mathcal{H}}_0$ with the set of the eigen-function $\psi_n^{(0)}$ and perturbation $\hat{V}$ one can construct perturbed wave-functions (see e.g. Landau & Lifshitz, vol. 3, § 38)

$$\psi_n^{(1)} = -\sum_{k\neq n} \frac{V_{kn}\psi_k^{(0)}}{E_k - E_n} \tag{12}$$

- To see that this is the same as Eq. (30), let us rewrite the expression

Green's function of the Schrödinger equation

$(\hat{\mathcal{H}}_0 + \hat{V})\psi = E\psi$ similarly to (28):

$$\psi = -\frac{1}{\hat{\mathcal{H}}_0 - E}\hat{V}\psi \tag{13}$$

- The object

$$\hat{G}_E \equiv \int dn \frac{|n\rangle \langle n|}{E_n - E} \tag{14}$$

(where $|n\rangle$ is the full set of eigen-functions of $\hat{\mathcal{H}}_0$) has a special meaning. Namely, consider a Hamiltonian $\mathcal{H}$. Then the following expression holds:

$$(\hat{\mathcal{H}} - E)\hat{G}_E = \int dn\ |n\rangle \langle n| = \mathbb{1} \tag{15}$$

i.e. this is **Green's function of the operator $\mathcal{H}$**

- In $|x\rangle$ representation

$$G_E(x, x') = \langle x| \hat{G}_E |x'\rangle = \int dn \frac{\psi_n(x)\psi_n^*(x')}{E_n - E} \tag{16}$$

and

$$(\hat{\mathcal{H}} - E)G_E(x, x') = \delta(x - x') \tag{17}$$

- **Demonstrate** that the Green's function of the free Schrödinger equation is

$$\text{Schrödinger} \quad G_E(\boldsymbol{x}, \boldsymbol{x}') = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}')}}{\frac{\boldsymbol{p}^2}{2m} - E} \tag{18}$$

Obtain it from definition (14) and (18) using $|n\rangle = |\boldsymbol{p}\rangle$ and taking into account that $\langle \boldsymbol{x}|\,\boldsymbol{p}\rangle = \frac{1}{\sqrt{V}}e^{i\boldsymbol{p}\cdot\boldsymbol{x}}$.

Green's function of the Dirac equation

- Similarly, the Green's function of the Dirac equation is defined in the Fourier space as[3]

$$\frac{1}{\hat{\mathcal{D}}^{(0)}} \equiv \tilde{S}_F(p) = \frac{1}{\gamma \cdot p - m} \tag{19}$$

- acting on an arbitrary function $\psi(x)$ as

$$\langle x| \frac{1}{\hat{\mathcal{D}}^{(0)}} |\psi\rangle = \int \frac{d^4p}{(2\pi)^4} \int d^4x' \frac{e^{-ip\cdot(x-x')}}{\gamma \cdot p - m} \psi(x') \tag{20}$$

- Notice, that unlike the previous (scalar) Green's function, this time we have a 4×4 matrix in spinor indices in the denominator of (19).

[3]Following Bjorken&Drell, we will use the notation S_F for the Green's function of the Dirac equation.

Green's function of the Dirac equation

- *The expression* $\frac{1}{\gamma\cdot p - m}$ *with matrices in the denominator should be understood in the following sense:*

$$\frac{1}{\gamma\cdot p - m} = \frac{(\gamma\cdot p + m)}{(\gamma\cdot p - m)(\gamma\cdot p + m)} = \frac{(\gamma\cdot p + m)}{p^2 - m^2} \tag{21}$$

- **Show** that the r.h.s. of (21) is just an inverse matrix of the 4×4 matrix $(\gamma\cdot p - m\mathbb{1})$.

- Green's function in real space:

$$S_F(x - x') = \int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x-x')}}{p^2 - m^2}(\gamma\cdot p + m) \tag{22}$$

Retarded/Advanced Green's functions

- When computing the Fourier transform of $\frac{1}{p_0^2 - \boldsymbol{p}^2 - m^2}$ we encounter a pole for all $p_0 = \pm\sqrt{\boldsymbol{p}^2 + m^2}$. (the same problem we actually had with the Green's function of the Schrödinger equation (18) — it has a pole on the real axis whenever $\frac{\boldsymbol{p}^2}{2m} = E$)

- The poles of the Green's function have deep physical meaning — they tell you what kind of particles exist in the theory.

- Recall, that the Green's function defines a solution of differential operator $\hat{L}$ with a δ-functional source. In our case

$$\hat{L}_x G(x, x') = \delta^{(4)}(x - x') \tag{23}$$

- Based on boundary conditions (in time) one can define **retarded**

Retarded/Advanced Green's functions

and **advanced** Green's functions:

$$G_{\text{Ret}}(x-x') = 0 \quad \text{if} \quad t < t'$$
$$G_{\text{Adv}}(x-x') = 0 \quad \text{if} \quad t > t' \tag{24}$$

- Their meaning is clear: if at the moment $t = 0$ in the point $\boldsymbol{x} = 0$ we turned on a perturbation ("hit a system with a hammer") then there is a wave propagating through the system for $t > 0$. On the other hand, for $t < 0$ there is nothing , and therefore $G_{\text{Ret}}(x - x') = 0$ for $t < t'$. (the advanced Green's function is simply anti-causal, demonstrating time-reversal symmetry)

- To define a retarded Green's function of the Schrödinger equation one adds a "prescription" of how to compute the integral (18):

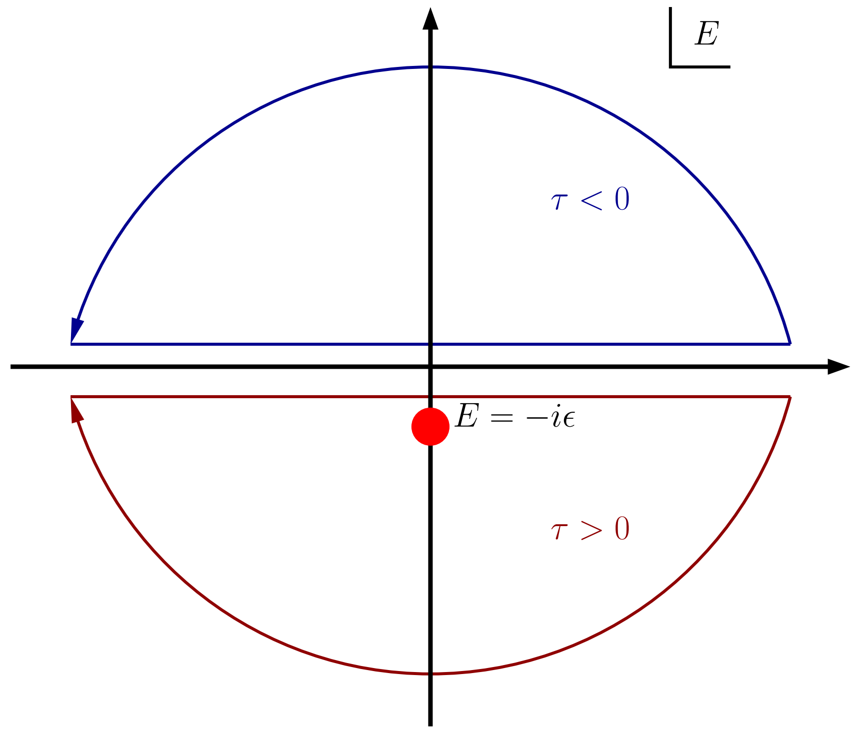
$$\text{Schrödinger} \quad G_{\text{Ret}}(x-x') = \int \frac{dE}{2\pi} e^{-iE(t-t')} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}')}}{\frac{\boldsymbol{p}^2}{2m} - E - \boldsymbol{i\epsilon}} \tag{25}$$

Retarded/Advanced Green's functions

- The symbol $i\epsilon$ should be understood as follows

$$\lim_{\epsilon\to+0}\int_{-\infty}^{\infty}\frac{dE}{2\pi i}e^{-iE\tau}\frac{1}{E+i\epsilon}=\theta(\tau) \tag{26}$$

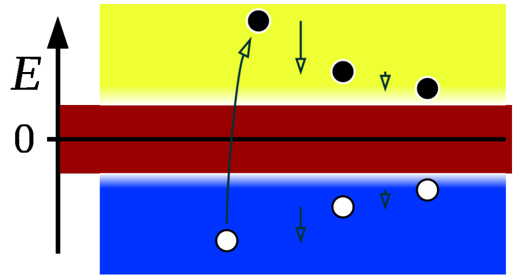
where $\theta(\tau)$ is step-function, the integral is performed in the complex plane $E = E' + iE''$ and contour is chosen depending on the sign of τ as shown in Fig.



- **Compute** the Green's function of non-relativistic particle in the $(t, \boldsymbol{x})$ space (i.e. compute the integrals in (25))

Causal Green's function

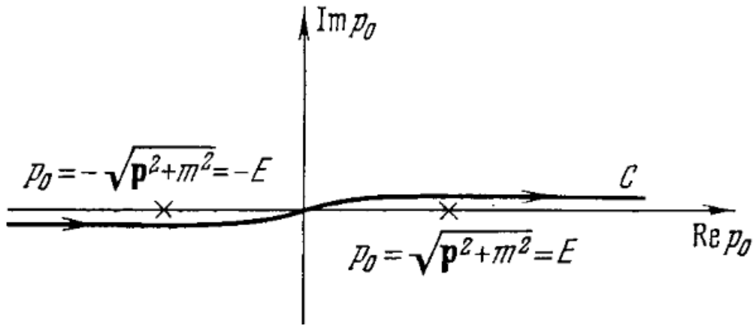
- In case of Dirac equation we have particles and anti-particles.
- Anti-particles (holes) are moving "backward in time" (when electron within the Dirac sea moves to the left, hole (empty space) moves to the right)
- **Recall:**



- Therefore, in case of the Dirac equation we expect to see **retarded** Green's function for particles ($E > 0$) and **advanced** Green's functions for anti-particles ($E < 0$)

Causal Green's function

- This is realized by the following $i\epsilon$ prescription



$$S_F(x-x') = \int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x-x')}}{p^2-m^2+i\epsilon}(\gamma\cdot p+m) \quad (27)$$

Iterative solution of the Dirac equation

- Consider the Dirac operator $\boxed{\hat{\mathcal{D}}^{(0)} \equiv \gamma \cdot \hat{p} - m}$ to which one adds a small perturbation $\hat{V}$ so that $\hat{\mathcal{D}} = \hat{\mathcal{D}}^{(0)} + \hat{V}$.

- A formal solution of the full Dirac equation is given by the following expression:

$$\psi = -\frac{1}{\hat{\mathcal{D}}^{(0)}} \hat{V} \psi \tag{28}$$

- Eq. (28) is just a formal rearrangement of $(\hat{\mathcal{D}}^{(0)} + \hat{V})\psi = 0$. However, written in this form, it allows for **the solution by iterations**

- Namely, a sequence of solutions $\{\psi^{(n)}\}$, $n = 0, 1, \ldots$ where

$$\psi^{(n)} = -\frac{1}{\hat{\mathcal{D}}^{(0)}} \hat{V} \psi^{(n-1)} \tag{29}$$

(and $\psi^{(0)}$ is the solution of the unperturbed Dirac equation $\hat{\mathcal{H}}^0$) will converge to the true solution of (28)

Green's function of the Dirac equation

- In practice one often restricts itself to first few iterations, assuming that the perturbation $\hat{V}$ is "small enough".

- **First order:**

$$\psi^{(1)} = -\frac{1}{\hat{\mathcal{D}}^{(0)}}\hat{V}\psi^{(0)} \tag{30}$$

- **Second order:**

$$\psi^{(2)} = \frac{1}{\hat{\mathcal{D}}^{(0)}}\hat{V}\frac{1}{\hat{\mathcal{D}}^{(0)}}\hat{V}\psi^{(0)} \tag{31}$$

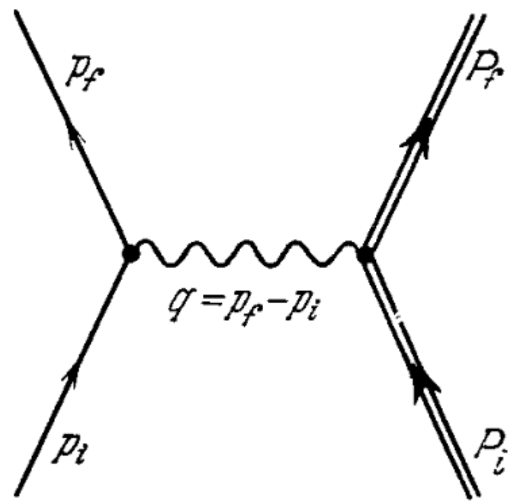
- **Recall** that in non-relativistic Quantum Mechanics if the transition $i \to f$ is forbidden, the analog of Eq. (1) reads (c.f. Landau & Lifshitz, vol. 3, § 43):

$$dw_{if} = \frac{2\pi}{\hbar}\left|\int dn\,\frac{V_{in}V_{nf}}{E_i - E_n}\right|^2 \delta(E_i - E_f)dn_f \tag{32}$$

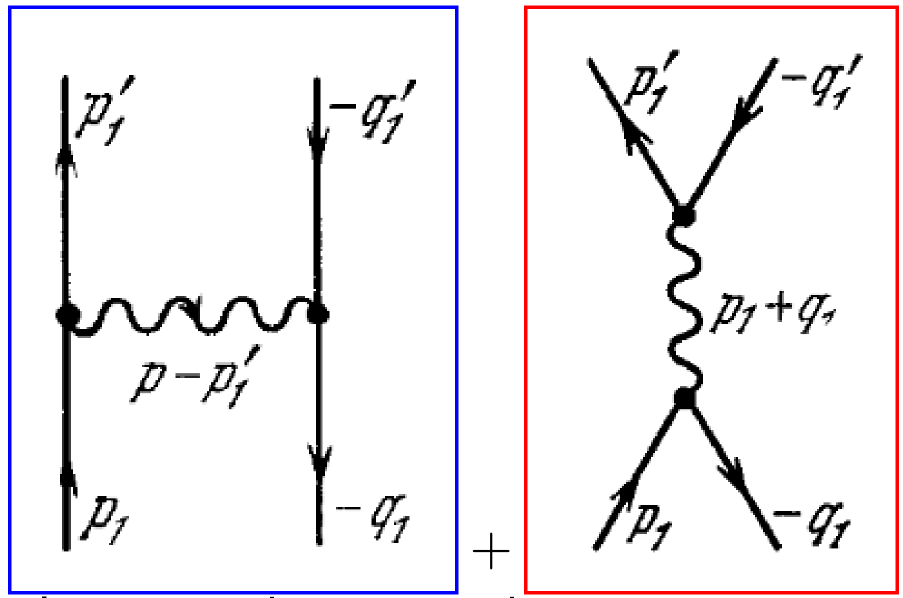
Green’s function of the Dirac equation

where $\int dn$ is the integral over any basis of **intermediate states**.

- Consider now electron-positron scattering.[4] Superficially it looks like electron-proton scattering we considered before (blue box).



Electron-proton scattering



Electron-positron scattering

- ... **However**, there is additional contribution to e^+e^- scattering. Although $e^- + e^+ \to \gamma$ is forbidden for real particles ($\langle e^-, e^+ | \hat{V} | \gamma \rangle = 0$) if both energy **and** momentum are conserved.

[4]See Bjorken & Drell, Sec. 7.9

Green's function of the Dirac equation

$$\hat{V} = \int d^3\boldsymbol{x}\, \left(\bar{\psi}(x)\gamma^\mu\psi(x)\right)\hat{A}_\mu(x) \tag{33}$$

- the second order expression *a la* Eq. (32) can be written:

$$dw_{if} = \frac{2\pi}{\hbar}\left|\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{(\bar{\psi}_i\gamma^\mu\psi_i)\,\langle 0|\,\hat{V}\,|\gamma_{\boldsymbol{k}}\rangle\,\langle\gamma_{\boldsymbol{k}}|\,\hat{V}\,|0\rangle\,(\bar{\psi}_f\gamma_\mu\psi_f)}{E_i-\omega_{\boldsymbol{k}}}\right|^2\delta(E_i{-}E_f)dn_f \tag{34}$$

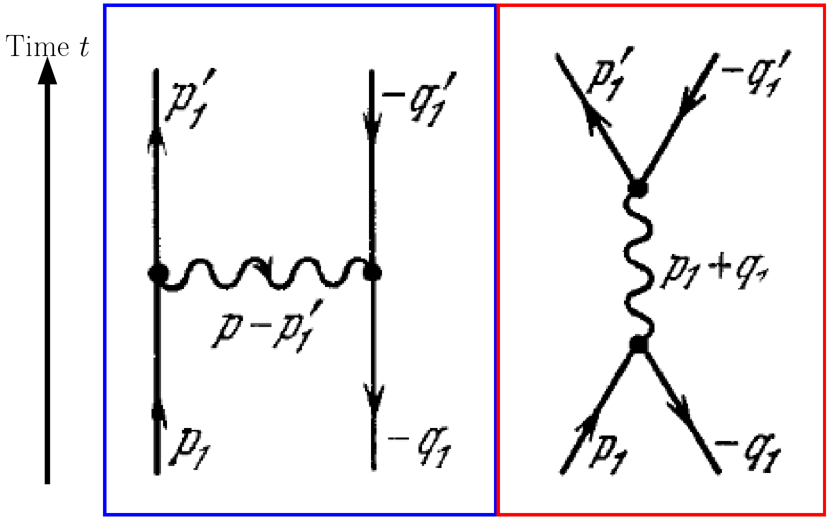
the sum is over **all photon intermediate states** with arbitrary 3-momentum $\boldsymbol{k}$.

- Amplitudes of two processes should be **added together** such that the probability of the process is proportional to $|\mathcal{M}_1+\mathcal{M}_2|^2$ rather than $|\mathcal{M}_1|^2+|\mathcal{M}_2|^2$ (interference terms are present).

$$|\mathcal{M}|^2 = (\dots)\left|\underbrace{i\frac{\bar{u}(p_1')(-i\gamma_\mu)u(p_1)\bar{v}(q_1)(-i\gamma^\mu)v(q_1')}{(p_1-p_1')^2}}_{\text{blue diagram}} - \underbrace{i\frac{\bar{u}(p_1')(-i\gamma_\mu)v(q_1')\bar{v}(q_1)(-i\gamma^\mu)u(p_1)}{(p_1+q_1)^2}}_{\text{red diagram}}\right|^2 \tag{35}$$

where ... is a prefactor, depending on energies/masses of particles.

Green's function of the Dirac equation

- 

  – "Blue" diagram can be considered as a propagation of a particle (say, electron) in the electromagnetic field, created by the other particle (positron)
  – "Red" diagram – electron-positron pair temporary disappears into a photon, photons "propagates" and the creates electron-positron pair

- If we consider time as going upwards, then for the blue diagram **at any given moment of time** there exists electron + positron. For the "red" diagram, for some period of time **only virtual photon exists**