Topics in Standard Model

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Consequences of the Dirac sea

- Presence of the negative-energy levels means that you can create particle-antiparticle pairs out of "nowhere"

- Particles in the pair can be real, but they can be also virtual (i.e. $E^2 - \boldsymbol{p}^2 \neq m^2$)

- According to the Heisenberg uncertainty relation $\Delta E\ \Delta t\ \gtrsim\ 1$, if one measures the state of system two times, separated by a short period $\Delta t \ll 1/m$, one will find a state with $1, 2, 3, \ldots$ additional pairs.

- It means that we no longer work with definite number of particles: number of particles may change! (Contrary to non-relativistic quantum mechanics)

- We need an approach that naturally takes into account states with different number of particles (we will return to this point in this Lecture)

Interaction of light with the Dirac sea

Since vacuum is not "empty", electromagnetic waves act on it non-trivially:

- the virtual particle-antiparticle pairs are excited
- the pairs are polarized by the electric field of the wave
- polarization changes the propapagation of the wave (**vacuum polarization**)

Two different electromagnetic waves can act on each other, through the interaction of the polarized virtual pairs. Light can scatter off light even in the vacuum!

See V. Dunne
1202.1557

The vacuum behaves like a medium.

Consequences of Dirac theory of positrons

- Photons are **bosons** (particles of spin = 1). Electrons/positrons are **fermions** particles of spin = 1/2. Therefore, angular momentum conservation means that **photon couples to electron + positron**

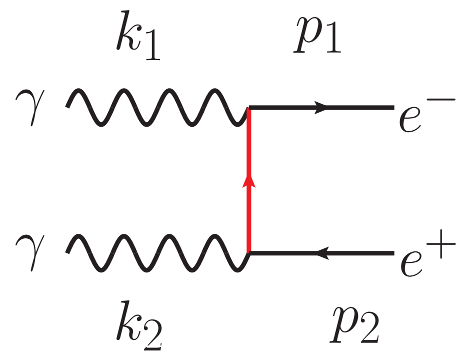
- Photons could produce electron-positron pairs. **However**, the process $\boxed{\gamma \to e^+ e^-}$ is not possible if all particle are "real" (i.e. photon obeys $E = cp$, electron/positron $E = \sqrt{p^2c^2 + m_e^2c^4}$ – "**on-shell conditions**")

- **Demonstrate** that based on kinematic considerations
  - photon **cannot** decay into an electron-positron pair (hint: hint use center-of-mass frame)
  - free electron **cannot** emit a photon
  - electron **can** emit a photon in the medium (i.e. speed of light $v < c$) — **Cherenkov effect**

Consequences of Dirac theory of positrons

- Instead, a **pair of photons** can produce electron-positron pair via

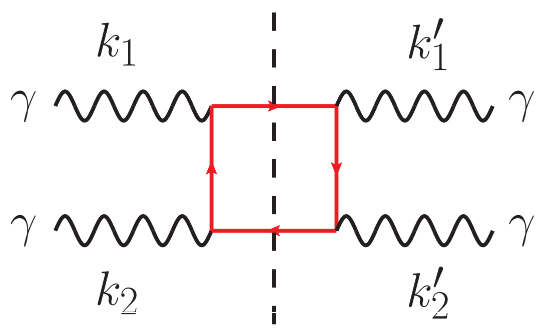
$$\boxed{\gamma + \gamma \to e^+ e^-}:$$



where $(k_1 + k_2)^2 \geq 4m_e^2$

- Similarly, electron-positron pair can **annihilate** into a pair of photons

- Kinematically, the red electron is **virtual** (i.e. for it $E \neq \sqrt{p^2c^2 + m^2c^4}$ – **check this**)

- 

If energies of incoming photons are smaller than twice the electron mass (i.e. $(k_1 + k_2)^2 < 4m_e^2$) photons produce only virtual electron-positron pair which can then "annihilate" into another pair of photons – **light-on-light scattering**

Example of a system with infinite number of particles – electromagnetic field

Electromagnetic field as collection of oscillators [1]

- Consider the solution of wave equation for vector potential $\vec{A}$ (impose $A_0 = 0$ and $\operatorname{div} \vec{A} = 0$)

$$\frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \Delta \boldsymbol{A} = 0 \qquad (1)$$

- Solution

$$\boldsymbol{A}(x,t) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\left[\boldsymbol{a}_{\boldsymbol{k}}(t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \boldsymbol{a}^*_{\boldsymbol{k}}(t)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right] \qquad (2)$$

where the complex functions $\boldsymbol{a}_{\boldsymbol{k}}(t)$ have the following time dependence

$$\boldsymbol{a}_{\boldsymbol{k}}(t) = \boldsymbol{a}_{\boldsymbol{k}} e^{-i\omega_{\boldsymbol{k}} t}, \qquad \omega_{\boldsymbol{k}} = |\boldsymbol{k}| \qquad (3)$$

[0] See Landau & Lifshitz, Vol. 4, §2

Generalized coordinate and momentum

- **Show** that
  - for any set of $\{\boldsymbol{a_k}\}$ expression (2) is the solution of the wave equation (2)
  - $\boldsymbol{a_k}(t)$ and $\boldsymbol{a^*_k}(t)$ obey the following equations:

$$\dot{\boldsymbol{a}}_{\boldsymbol{k}}(t) = -i\omega_{\boldsymbol{k}}\boldsymbol{a}_{\boldsymbol{k}}(t) \quad , \quad \dot{\boldsymbol{a}}^*_{\boldsymbol{k}}(t) = i\omega_{\boldsymbol{k}}\boldsymbol{a}^*_{\boldsymbol{k}}(t) \tag{4}$$

- Notice that we re-wrote partial differential equation (2) second order in time into a set of ordinary differential equations for infinite set of functions $\boldsymbol{a_k}(t)$ and $\boldsymbol{a^*_k}(t)$. **Any solution of the free Maxwell's equation** is parametrized by the (infinite) set of complex number $\{\boldsymbol{a_k}, \boldsymbol{a^*_k}\}$

- To make the meaning of Eqs. (4) clear, let us introduce their imaginary and real parts.

$$\left.\begin{array}{l} \boldsymbol{Q_k} \equiv \boldsymbol{a_k} + \boldsymbol{a^*_k} \\ \boldsymbol{P_k} \equiv -i\omega_{\boldsymbol{k}}\big(\boldsymbol{a_k} - \boldsymbol{a^*_k}\big) \end{array}\right\} \overset{\text{dynamics}}{\Longrightarrow} \left\{\begin{array}{l} \dot{\boldsymbol{Q}}_{\boldsymbol{k}} = \boldsymbol{P_k} \\ \dot{\boldsymbol{P}}_{\boldsymbol{k}} = -\omega^2_{\boldsymbol{k}}\boldsymbol{Q_k} \end{array}\right. \tag{5}$$

Hamiltonian of electromagnetic field

- Hamiltonian (total energy) of electromagnetic field is given by

$$\mathcal{H} = \frac{1}{2}\int d^3\boldsymbol{x}\left[\boldsymbol{E}^2 + \boldsymbol{B}^2\right] = \frac{1}{2}\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\left[\boldsymbol{E}_{\boldsymbol{k}}^2 + \boldsymbol{B}_{\boldsymbol{k}}^2\right] \qquad (6)$$

- Using mode expansion (2) and definition (5) we can write with frequencies $\omega_{\boldsymbol{k}}$

$$\mathcal{H}[\boldsymbol{Q}_{\boldsymbol{k}}, \boldsymbol{P}_{\boldsymbol{k}}] = \frac{1}{2}\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\left[\boldsymbol{P}_{\boldsymbol{k}}{}^2 + \omega_{\boldsymbol{k}}^2 \boldsymbol{Q}_{\boldsymbol{k}}{}^2\right] \qquad (7)$$

- Therefore dynamical equations (5) are nothing by the Hamiltonian equations

$$\dot{\boldsymbol{Q}}_{\boldsymbol{k}} = \frac{\partial \mathcal{H}}{\partial \boldsymbol{P}_{\boldsymbol{k}}} \quad , \quad \dot{\boldsymbol{P}}_{\boldsymbol{k}} = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{Q}_{\boldsymbol{k}}} \qquad (8)$$

with Hamiltonian (7)

Hamiltonian of electromagnetic field

- Eqs. (7)–(8) describe Hamiltonian dynamics of a sum of independent oscillators with frequencies $\omega_{\boldsymbol{k}}$

Classical electromagnetic field can be considered as an infinite sum of oscillators with frequencies $\omega_{\boldsymbol{k}}$

Quantum mechanical oscillator

- **Recall:** for quantum mechanical oscillator, described by the Hamiltonian

$$\hat{\mathcal{H}}_{\text{osc}} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{m\omega^2}{2}x^2 \tag{9}$$

one can introduce **creation and annihilation operators**:

$$\hat{a}^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega x + \hbar\partial_x) \quad ; \quad \hat{a} = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega x - \hbar\partial_x) \tag{10}$$

- Commutation $[\hat{a}, \hat{a}^\dagger] = 1$

- Hamiltonian can be rewritten as $\hat{\mathcal{H}}_{\text{osc}} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$

Properties of creation/annihilation operators

- Commutation $[\hat{a}, \hat{a}^\dagger] = 1$
- If one defines a **vacuum** $|0\rangle$, such that $\hat{a}\,|0\rangle = 0$ (**Fock vacuum**) then a state $|n\rangle \equiv (\hat{a}^\dagger)^n\,|0\rangle$ is the eigenstate of the Hamiltonian (9) with $E_n = \hbar\omega(n + \frac{1}{2})$, $n = 0, 1, \ldots$
- Given $|n\rangle$, $n > 0$, $\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$ and $\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle$
- Time evolution of the operators $\hat{a}, \hat{a}^\dagger$:

$$i\hbar\frac{\partial\hat{a}}{\partial t} = [\mathcal{H}_{\text{osc}}, \hat{a}] \tag{11}$$

and Hermitian conjugated for $\hat{a}^\dagger$

Birth of quantum field theory

- Dirac (1927) proposes to treat radiation as a collection of **quantum** oscillators

Paul A.M. Dirac ***Quantum theory of emission and absorption of radiation***

Proc.Roy.Soc.Lond. A114 (1927) 243

- Take the classical solution (2)

$$\boldsymbol{A}(x,t) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\left[\boldsymbol{a}_{\boldsymbol{k}}(t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \boldsymbol{a}^*_{\boldsymbol{k}}(t)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right]$$

- Introduce creation/annihilation operators $\hat{a}_{\boldsymbol{k}}$, $\hat{a}^\dagger_{\boldsymbol{k}}$

$$[\hat{a}_{\boldsymbol{k}}, \hat{a}^\dagger_{\boldsymbol{p}}] = \hbar\delta_{\boldsymbol{k},\boldsymbol{p}} \qquad [\hat{a}_{\boldsymbol{k}}, \hat{a}_{\boldsymbol{p}}] = 0 \tag{12}$$

Birth of quantum field theory

- Replace Eq. (2) with a quantum operator

$$\hat{\boldsymbol{A}}(x,t) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\left[\hat{\boldsymbol{a}}_{\boldsymbol{k}}(t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \hat{\boldsymbol{a}}^\dagger_{\boldsymbol{k}}(t)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right] \tag{13}$$

- Operator $\boxed{\hat{a}^\dagger_{\boldsymbol{k}}}$ **creates photon with momentum k and frequency $\omega_{\boldsymbol{k}}$**

- Operator $\boxed{\hat{a}_{\boldsymbol{k}}}$ **destroys photon with momentum k and frequency $\omega_{\boldsymbol{k}}$** (if exists in the initial state)

- State without photons $\leftrightarrow$ Fock vacuum:

$$\hat{a}_{\boldsymbol{k}}\left|0\right\rangle = 0 \quad \forall \boldsymbol{k} \tag{14}$$

Birth of quantum field theory

- State with N photons with momenta $\boldsymbol{k}_1, \boldsymbol{k}_2, \ldots, \boldsymbol{k}_N$:

$$|\boldsymbol{k}_1, \boldsymbol{k}_2, \ldots, \boldsymbol{k}_N\rangle = \hat{\boldsymbol{a}}^\dagger_{\boldsymbol{k}_1} \hat{\boldsymbol{a}}^\dagger_{\boldsymbol{k}_2} \ldots \hat{\boldsymbol{a}}^\dagger_{\boldsymbol{k}_N} |0\rangle \qquad (15)$$

- The interaction of matter with electromagnetic radiation is given by

$$\hat{V}_{\text{int}} = \int d^4x \, \hat{\jmath}^\mu(x) \hat{A}_\mu(x) \qquad (16)$$

where $\hat{\jmath}^\mu(x)$ is an operator, describing matter (e.g. atom) and $\hat{A}_\mu$ is given by the analog of (13)

- This model allowed Dirac to compute absorption/emission of radiation by atoms

- **Spontaneous emission** means that we should compute $\langle 1| \hat{A}_\mu(x) |0\rangle$ matrix element and multiply it by $\langle f| \hat{\jmath}^\mu(x) |i\rangle$. Using (13) this gives

Birth of quantum field theory

probability

$$dP \propto |V_{fi}|^2 \delta(E_i - E_f - \omega) \frac{d^3k}{(2\pi)^3} \tag{17}$$

where $V_{fi} = \int d^3x \, e^{i\boldsymbol{k}\cdot\vec{x}} \langle f| \hat{\jmath}^\mu(x) |i\rangle$

- **Induced emission** means that we should compute $\langle N+1| \hat{A}_\mu(x) |N\rangle$ matrix element and multiply it by $\langle f| \hat{\jmath}^\mu(x) |i\rangle$

- Using these results one can demonstrate for example the Einstein's relation between coefficients of emissions and absorption of an atom

$$\frac{P_{\text{emis}}}{P_{\text{abs}}} = \frac{N+1}{N} \tag{18}$$

where N – number of photons in the **initial state**

Quantum electrodynamics (QED) – first quantum field theory has been created. Free fields with interaction treated **perturbatively** in *fine-structure constant:* $\alpha = \frac{e^2}{\hbar c}$

Perturbation theory

Electron scattering in Coulomb field [2]

- In non-relativistic quantum mechanics if Hamiltonian has the form $\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \hat{V}$ then the probability of transition between an initial state $\psi_i(x)$ and the final state $\psi_f(x)$ of **unperturbed** Hamiltonian $\hat{\mathcal{H}}_0$ is given by (Landau & Lifshitz, vol. 3, § 43):

$$dw_{if} = \frac{2\pi}{\hbar}|V_{if}|^2\delta(E_i - E_f)dn_f \qquad (19)$$

  where $|V_{if}|$ is the matrix element between initial and final states; and dn_f is the number of final states with the energy E_f (degeneracy of the energy level).

- In the case of Dirac equation, the interaction is given by

$$V_{\text{int}} = \int d^4x\, \bar{\psi}(x)\gamma^\mu A_\mu(x)\psi(x) \qquad (20)$$

  recall that electric current $j^\mu = \bar{\psi}(x)\gamma^\mu\psi(x)$

[1] Following Bjorken & Drell, Sec. 7.1

Electron scattering in Coulomb field [4]

- If we consider static point source with the Coulomb field

$$A_0(\boldsymbol{x}) = \frac{Ze}{4\pi|\boldsymbol{x}|} \tag{21}$$

and wave-functions[3]

$$\psi_i(x) = \sqrt{\frac{m}{E_i V}} u_s(p_i) e^{-ip_i\cdot x} \quad , \quad \bar{\psi}_f(x) = \sqrt{\frac{m}{E_f V}} \bar{u}_r(p_f) e^{ip_f\cdot x} \tag{22}$$

($E_i = E_f$)

- Following (19) we write the matrix element

$$V_{if} = \frac{Ze^2}{4\pi}\frac{1}{V}\sqrt{\frac{m^2}{E_i E_f}} \bar{u}_r(p_f)\gamma^0 u_s(p_f) \int d^3\boldsymbol{x} e^{i\boldsymbol{x}\cdot(\boldsymbol{p}_i - \boldsymbol{p}_f)} A_0(\boldsymbol{x}) \tag{23}$$

[3] Here u_s, $\bar{u}_r$ are 4-component spinors – solution of the Dirac equations $(\gamma \cdot p - m)u_s = 0$, $\bar{u}_r(\gamma \cdot p + m) = 0$, $s = \pm$, $r = \pm$ – polarizations of spin.

Electron scattering in Coulomb field [5]

- Degeneracy of a final state with E_f is given by

$$dn_f = \mathbf{2} \times \underbrace{\int\limits_{p_0>0} d^4p\,\delta(p^2-m^2)}_{\text{density of states}} = \frac{d^3\boldsymbol{p}_f}{(2\pi)^3 E_f} \tag{24}$$

- As a result we get

$$\begin{aligned} dw_{if} &= 2\pi |V_{if}|^2 \frac{d^3\boldsymbol{p}_f}{(2\pi)^3 E_f} \\ &= \frac{Z^2\,(4\pi\alpha)^2 m^2}{E_i V} \frac{|\bar{u}_r(p_f)\gamma^0 u_s(p_f)|^2}{|\boldsymbol{p}_i - \boldsymbol{p}_f|^4} \frac{d^3\boldsymbol{p}_f}{(2\pi)^3 E_f}\delta(E_i - E_f) \end{aligned} \tag{25}$$

Sum over spins

- Our formulas contained spinors $u_s(p_i)$ and $\bar{u}_r(p_f)$ where indexes $s, r = \pm$ – spin polarizations

- In real experiments, the detectors usually do not distinguish polarizations of particles in final state. Therefore, we measure the probabilities, **summed** over polarization states $s = +$ and $s = -$.

- The polarizations of the initial particles are not fixed usually, as well. Instead, there are equal amounts of both spin states, and effectively we measure the half of reactions with $r = +$, and the other half of reactions with $r = -$. To take it into account in our formulas, we have to **average** $|V_{fi}|^2$ over the initial polarization states.

- The useful identity (**the spin sum rule**)

$$\sum_{s=\pm} u_s(p)\bar{u}_s(p) = p_\mu \gamma^\mu + m, \tag{26}$$

Sum over spins

- **Show** that Eq. (26) becomes identity if you use on it with the Dirac equation on the left or on the right
- **Derive** Eq. (26) in the rest-frame of a massive particle (i.e. $p = (E, \vec{0})$).
- **Derive** Eq. (26) for a massless particle (i.e. $p^\mu = (cp_z, 0, 0, p_z)$).

■ Together with the rearrangement

$$|\bar{u}_r \gamma^0 u_s|^2 = \bar{u}_r \gamma^0 u_s \bar{u}_s \gamma^0 u_r = \mathrm{Tr}\left[u_r \bar{u}_r \gamma^0 u_s \bar{u}_s \gamma^0\right] \tag{27}$$

the spin sum rule leads to

averaging over initial polarizations

$$\frac{1}{2}\sum_s \sum_r |\bar{u}_r \gamma^0 u_s|^2 = 2(E_i E_f + \boldsymbol{p}_i \boldsymbol{p}_f + m^2) \tag{28}$$

summing over the final polarizations

Probability in unit time

- The δ-function of energy that appears in Eq. (25) should be understood in the following sense.

$$\delta(E_i - E_f) = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} e^{i(E_i - E_f)t} dt \tag{29}$$

- However, in real experiments we are interested in finite intervals of time T, so

$$\int\limits_{-\infty}^{+\infty} e^{i(E_i - E_f)t} dt \approx \int\limits_{-T/2}^{+T/2} e^{i(E_i - E_f)t} dt = 2\frac{\sin\left(\frac{T}{2}(E_f - E_i)\right)}{E_f - E_i} \tag{30}$$

- The delta function is now replaced by the function localized around $E_i = E_f$. The width of the localization is $\sim 1/T$, corresponding to the Heisenberg uncertainty relation $T\Delta E \gtrsim 1$.

Probability in unit time

- The product of delta functions $\delta(E_i - E_f)\delta(E_i - E_f)$ may be replaced by

$$\delta(E_i - E_f)\delta(0) = \delta(E_i - E_f)\frac{T}{2\pi} \tag{31}$$

- dw_{if} is the probability that the reaction happens during the whole interval of time T, hence $dw_{if} \propto T$. It is more convenient to consider the probability of interaction **per unit time**, that does not depend on T,

$$\frac{dw_{if}}{T} \tag{32}$$

  This quantity involves only one delta-function, not two of them, as it was before.

Electron scattering in Coulomb field [8]

- The quantity dw_{if} is not measured directly. Rather in experiemnts one measures **differential cross-section**

  Express the cross-section in terms of dw_{if}

- The probability dw_{if} still depends on flux of incident particles (that is the number of particles crossing unit area per unit time), that is given by $\vec{\jmath} = \bar{\psi}_i \vec{\gamma} \psi_i$.

- **Show** that the differential cross-section has the form

$$\frac{d\sigma}{d\Omega} = \int\limits_{p_f} \frac{dw_{if}}{T} \frac{1}{|\vec{\jmath}|} = 8 \frac{Z^2 \alpha^2 m^2}{|\boldsymbol{p}_i - \boldsymbol{p}_f|^4} (E_i E_f + \boldsymbol{p}_i \boldsymbol{p}_f + m^2) \tag{33}$$

- In the non-relativistic limit, the cross-section reduces to the Rutherford formula[6]

[6] Show it.

Electron scattering in Coulomb field [9]

- The cross-section is singular for the forward scattering of electron, when $\boldsymbol{p}_f = \boldsymbol{p}_i$. It is the same type of singularity, that one finds for the Rutherford scattering, due to the long-range nature of the Coulomb force.[7]

[7] Recall, that in classical mechanics the singularity corresponds to the scattering of charged particles with large impact parameter. Even these large impact parameters are important for the cross section, because the Coulomb field decays slowly with distance.

Electron scattering on proton

- Consider next the situation when the electromagnetic field is created by other particle ("proton")

- While the formulas (22)–(24) remain true, the expression for A_μ changes.

- If proton is described by a spinor Ψ, then its electric current is

$$J^\mu(y) = \bar{\Psi}(y)\gamma^\mu\Psi(y) \tag{34}$$

  (the form of Ψ_i and $\bar{\Psi}_f$ is the same as Eq. (22) with $m \to M_p$ and different momenta)

- The electromagnetic field obeys the Klein-Gordon equation

$$\Box A_\mu = J_\mu \quad \text{or} \quad A_\mu(x) = \frac{1}{\Box} J_\mu(y) \tag{35}$$

Electron scattering on proton

- The operator $\Box^{-1}$ (inverse to the Klein-Gordon operator) can be easily constructed if one considers (35) in Fourier space:[10]

$$p^2\tilde{A}_\mu(p) = \tilde{J}_\mu(p) \quad \text{or} \quad \tilde{A}_\mu(p) = \frac{1}{p^2}\tilde{J}_\mu(p) \tag{36}$$

- Therefore, the solution of Eq. (35) with arbitrary source term is

$$A_\mu(x) = \int \frac{d^4p}{(2\pi)^4}\, e^{ip\cdot x}\frac{\tilde{J}_\mu(p)}{p^2} = \int \frac{d^4p}{(2\pi)^4}\int d^4y\, \frac{e^{ip\cdot(x-y)}}{p^2}J_\mu(y) \tag{37}$$

- Notice that in Eq. (23) we only need $\tilde{A}_\mu(p_i - p_f)$. The resulting expression is then equivalent to (25) if one substitutes

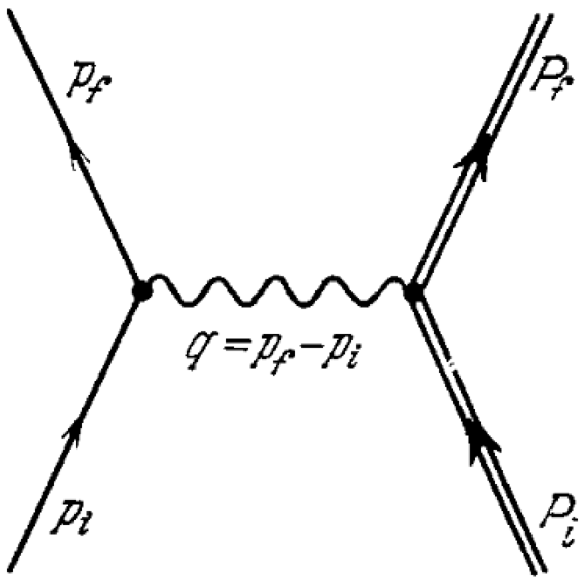
$$\gamma^0\frac{Z}{|\boldsymbol{q}|^2} \to \gamma^\mu\frac{1}{q^2}\sqrt{\frac{M_p^2}{E_i^{(p)}E_f^{(p)}}}\bar{U}_r(P_i)\gamma_\mu U_s(P_f)$$

where $q = p_f - p_i = P_i - P_f$ – transferred 4-momentum and $\boldsymbol{q}$ is its spatial component. 4-spinors U_s and $\bar{U}_r$ are the in- and out- 4-spinors of a proton..

[10]We denote by $\tilde{A}_\mu(p)$ and $\tilde{J}_\mu(p)$ Fourier transform

Electron scattering on proton

- That is the result looks like a scattering of electron in external field (25) where the external field A_μ is the field created by the proton (37) and (34).



So far we have considered two processes

  – Electron scatters on static electric field

  – Electron scatters on dynamic electromagnetic field, created by another moving particle (proton).

- What changes if instead of electromagnetic field we are taking **real photons**?