Topics in Standard Model

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Natural units

A slight detour...

- In particle physics it is convenient to work in the **"natural units"**:
  $$\boxed{\hbar = c = k_B = 1}$$

- This means that
  - [Distance] $=$ [Time]($\times$ c)
  - [Energy] $=$ [Mass]$(\times c^2)$ $=$ Momentum$(\times c)$ $=$[Temperature]$(\times k_B)$
  - [Distance] $=$ [Energy]$^{-1}$

- **Exercises: try to convert**:
  - Velocity $v = 0.1$ in cm/sec
  - Temperature = $1\,\mathrm{eV}$; $1\,\mathrm{keV}$; $1\,\mathrm{MeV}$; $1\,\mathrm{GeV}$ in Kelvins
  - Distance $1\,\mathrm{GeV}^{-1}$ in cm
  - 1 second in GeV^{-1}
  - $1\,\mathrm{gram}$ in GeV
  - $1\,\mathrm{g/cm}^3$ in GeV^4 and in $\mathrm{GeV/cm}^3$

- The Dirac equation

$$\left(i\frac{\partial}{\partial x^\mu}(\gamma^\mu)^\alpha_\beta - m\,\delta^\alpha_\beta\right)\psi^\beta = 0 \tag{1}$$

involves 4 Dirac matrices γ^μ (index $\mu = 0,1,2,3$), each matrix having size 4×4 (indices α,β run over their dimensions)[1]

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}; \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{2}$$

- Therefore, the wave-function ψ^β has actually 4 components

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \tag{3}$$

[1] We use the same representation, as in Lecture 1

and is called **spinor** (or 4-component spinor)

- Recall: if we square this equation we obtain Klein-Gordon equation for each component:

$$(\Box - m^2)\psi^\beta = 0 \quad \forall\, \beta \tag{4}$$

Probability and current density

- As planned, we have constructed an equation of the form $(i\partial_t - H)\psi = 0$
- Therefore the quantity

$$\int d^3x\, \psi^+\psi \tag{5}$$

  is conserved

- The quantity

$$\psi^\dagger\psi = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2 \geq 0 \tag{6}$$

  can be interpreted as the **probability density**. Contrary to the Klein-Gordon case, it is non-negative by construction.

- The density $\psi^\dagger\psi$ is a part of the current vector

$$j^\mu = \psi^\dagger\gamma^0\gamma^\mu\psi \tag{7}$$

Probability and current density

- As a consequence of the Dirac equation, this current is conserved[2]

$$\partial_\mu j^\mu = 0 \tag{8}$$

therefore the spatial part of j^μ has the meaning of the current density.

[2] Check this

Interaction with electromagnetic field [3]

- In the same paper [Proc. R. Soc. Lond. (1928) 610, 24] Dirac introduced coupling of spinors to electromagnetic field

- Recall in non-relativistic quantum mechanics coupling to the electromagnetic field was via "minimal coupling" (i.e. the momentum $\hat{p}_\mu \to \hat{P}_\mu = (\hat{p}_\mu - eA_\mu)$, P_μ is sometimes called "generalized momentum")

$$\frac{\hat{p}^2}{2m} \to \frac{1}{2m}\left(\hat{p} - e\vec{A}\right)^2 + eA_0(x) \tag{9}$$

- In Dirac equation, we make similar substitution

$$(\hat{p}_\mu\gamma^\mu - eA_\mu\gamma^\mu - m)\psi = 0 \tag{10}$$

- **Problem:** check that the Dirac equation (10) is invariant under the gauge transformation $A_\mu \to A_\mu - \partial_\mu f,\ \psi \to \psi e^{ief}$.

[2]Bjorken-Drell, Chap. 1, Sec. 1.4

Interaction with electromagnetic field. Problems

- **Write** the Dirac Hamiltonian in the constant magnetic field pointing in the z direction.
- Find the energy levels ("Landau levels") of this Hamiltonian.
- Compare them with non-relativistic Landau levels
  (see e.g. Landau & Lifshitz, vol. 3, § 112)

Lorentz transformation of the Dirac equation

- Free Dirac equation is linear and describes translational invariant theory $\Rightarrow$Let us look for the plane-wave solutions of the Dirac equation[4]

$$\psi(x) = u(p)e^{-ipx}, \quad px \equiv p^\mu x_\mu \equiv Et - \boldsymbol{px} \tag{11}$$

- Such a Fourier transform of the Dirac equation $(i\partial_\mu\gamma^\mu - m)\psi = 0$ becomes

$$(p_\mu\gamma^\mu - m)\psi = 0 \tag{12}$$

- If we change a reference frame 4-momentum **changes**

$$p_\mu \to \tilde{p}_\mu = \Lambda_\mu^{\ \nu}\, p_\nu \tag{13}$$

where $\Lambda_\mu^{\ \nu}$ is a 4×4 matrix of Lorentz transformation.

[4] **From now on**, to make notations simpler, we will use **bold** letters on 3-dimensional quantities instead of drawing vectors. So $\vec{A} \to \boldsymbol{A}$, coordinates $\vec{x} \to \boldsymbol{x}$ and momentum $\vec{p} \to \boldsymbol{p}$. Pauli matrices $\boldsymbol{\sigma} = \{\sigma_x, \sigma_y, \sigma_z\}$.

- **show** that the matrix

$$\Lambda = \begin{pmatrix} \cosh\omega & -\sinh\omega & 0 & 0 \\ \sinh\omega & \cosh\omega & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{14}$$

Defines a boost along the x axis with velocity v such that $\tanh\omega = \frac{v}{c}$

- We could introduce a new set of γ-matrices, defined via

$$\tilde{\gamma}^\mu = (\Lambda^{-1})^\mu{}_\nu \gamma^\nu \tag{15}$$

so that

$$\tilde{\gamma}^\mu \tilde{p}_\mu = \gamma^\mu p_\mu \tag{16}$$

- **Check** (16) based on the definition (15).

Show that the new set of $\tilde{\gamma}$ also satisfies $\{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\} = 2\eta^{\mu\nu}$

- As a consequence of (16) the same spinor $u(p)$ satisfied the transformed equation

$$(\tilde{p}_\mu \tilde{\gamma}^\mu - m)u(\tilde{p}) = 0, \tag{17}$$

where $u(\tilde{p}) = u(p(\tilde{p}))$ and p is expressed via $\tilde{p}$ by inverse of Eq. (13)

- However, if in the original frame of reference the form of the Dirac equation was

$$\begin{pmatrix} E - m & -\boldsymbol{p}\cdot\boldsymbol{\sigma} \\ \boldsymbol{p}\cdot\boldsymbol{\sigma} & -E - m \end{pmatrix} u = 0 \tag{18}$$

then after the Lorentz transformation the Eq. (18) would have a different form.

- Instead, we would prefer to work with the same equation, i.e. to have

$$\begin{pmatrix} \tilde{E} - m & -\tilde{\boldsymbol{p}}\cdot\boldsymbol{\sigma} \\ \tilde{\boldsymbol{p}}\cdot\boldsymbol{\sigma} & -\tilde{E} - m \end{pmatrix} \tilde{u}(\tilde{p}) = 0 \tag{19}$$

instead of Eq. (18).

- In order to get the same form of equation (with exactly the same set of γ-matrices) the spinor $u(p)$ should be modified (U is a 4×4 matrix)

$$\tilde{u} = Uu \tag{20}$$

- The matrix U should possess the following property:

$$(\gamma^\mu \tilde{p}_\mu - m)Uu = U(\tilde{\gamma}^\mu \tilde{p}_\mu - m)u \Longrightarrow \boxed{U^{-1}\gamma^\mu U = \tilde{\gamma}^\mu} \qquad (21)$$

- It can be shown[5] that the property (21) is satisfied by

$$U = \exp\left(\frac{i}{4}\phi_{\mu\nu}\sigma^{\mu\nu}\right) \qquad (22)$$

where $\phi_{\mu\nu}$ is an anti-symmetric 4×4 matrix parametrizing 6 Lorentz transformations (3 boosts $+$ 3 spatial rotations)

- Such multiplication mixes all 4 components of spinor

- 4-component spinor u changes in a different way than 4-vectors

[5] See Bjoren & Drell, Sec. 2.2; Peskin & Schroeder, Sec. 3.2. See the original Dirac's paper (§3) for a different proof of this fact.

Lorentz transformation of the Dirac equation

- **Show** that there are 16 matrices 4×4 **linearly independent** matrices that one can build from γ-matrices by multiplication

- Divide the spinor $u(p)$ in two **two-component spinors**

$$u = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \tag{23}$$

- The Dirac equation (18) then becomes:

$$\begin{cases} (E-m)\chi_1 - (\boldsymbol{p}\cdot\boldsymbol{\sigma})\chi_2 = 0 \\ (\boldsymbol{p}\cdot\boldsymbol{\sigma})\chi_1 - (E+m)\chi_2 = 0 \end{cases} \tag{24}$$

- Express χ_2 via χ_1 from the second line of Eq. (24)

$$\chi_2 = \frac{\boldsymbol{p}\cdot\boldsymbol{\sigma}}{E+m}\chi_1 \tag{25}$$

[5]Bjorken-Drell, Chap. 1, Sec. 1.4; See also the original paper by Dirac, Proc. R. Soc. Lond. (1928) **610**, 24

- If we plug (25) into the first line of Eq. (24), we will see that each of the spinors obeys Klein-Gordon equation:

$$(E^2 - \boldsymbol{p}^2 - m^2)\chi_{1,2}(p) = 0,$$

so both positive and negative energy solutions exist:

$$E = \pm\sqrt{\boldsymbol{p}^2 + m^2} \tag{26}$$

- Consider positive energies, $E > 0$. In the non-relativistic limit, $E - m \ll m$ and $|\vec{p}| \ll m$ we can write

$$\chi_2 \approx \frac{\boldsymbol{p}\cdot\boldsymbol{\sigma}}{2m}\chi_1 \tag{27}$$

(we neglected v^2/c^2 terms in Eq. (27)).

- As a result the equation for χ_1 then becomes just **the Schrödinger**

equation for a two-component spinor that can be re-written as

$$i\frac{\partial\chi_1}{\partial t} = m\chi_1 + \frac{(\boldsymbol{p}\cdot\boldsymbol{\sigma})(\boldsymbol{p}\cdot\boldsymbol{\sigma})}{2m}\chi_1 = \left(m + \frac{\boldsymbol{p}^2}{2m}\right)\chi_1 \tag{28}$$

- Introducing a trivial time dependence $\chi_1 \to e^{-imt}\chi_1'$, we find for χ_1' the ordinary Schrödinger
- **Derive Pauli Hamiltonian**

$$i\frac{\partial\chi_1}{\partial t} = \left[\frac{1}{2m}\left(\hat{\boldsymbol{p}} - e\boldsymbol{A}\right)^2 - \frac{e}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{B} + eA_0\right]\chi_1 \tag{29}$$

as a non-relativistic limit of the Dirac equation (10) with electromagnetic field. As A_μ depends on x, one cannot do the Fourier transformation and a differential operator will be acting on χ_1 to obtain χ_2

Spin and degrees of freedom

- By applying the rotations in 3-dimensional space (x, y, z) to χ_1, we extract the operator of total **angular momentum**[9]

$$\boldsymbol{J} = \boldsymbol{x} \times \boldsymbol{p} + \frac{1}{2}\boldsymbol{\sigma} \tag{30}$$

- The first term on the right-hand side is the operator of orbital angular momentum $\boldsymbol{L}$, the second term is therefore the spin operator $\boldsymbol{S}$.

- It means that for the state

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \boldsymbol{S}\,\chi_1 = \left(+\frac{1}{2}\right)\chi_1 \tag{31}$$

Therefore, this state has definite projection of spin on axis z, $s_z = +1/2$.

[9] See e.g Peskin & Schroeder, Sec. 3.5

Spin and degrees of freedom

In analogy, for the state

$$\chi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{S}\,\chi_1 = \left(-\frac{1}{2}\right)\chi_1. \tag{32}$$

In total, positive-energy solutions have **two degrees of freedom**, that correspond to spin-$1/2$ particles with definite spin projections.

- Now consider states with negative energy

$$E = -\sqrt{\boldsymbol{p}^2 + m^2} \tag{33}$$

- The system is unstable, since it will try to choose the state with the lowest possible energy, but there is no natural lower bound on the value of the negative energy.

- The interpretation of Dirac: Since fermions obey the Pauli principle, the occupation number of each energy level cannot exceed 1[10]

- Assume that we start from the many-fermion system, where all the energy states with $E < 0$ are fully occupied (the Dirac sea).

[9] See Bjoren & Drell, Sec. 5.1–5.3

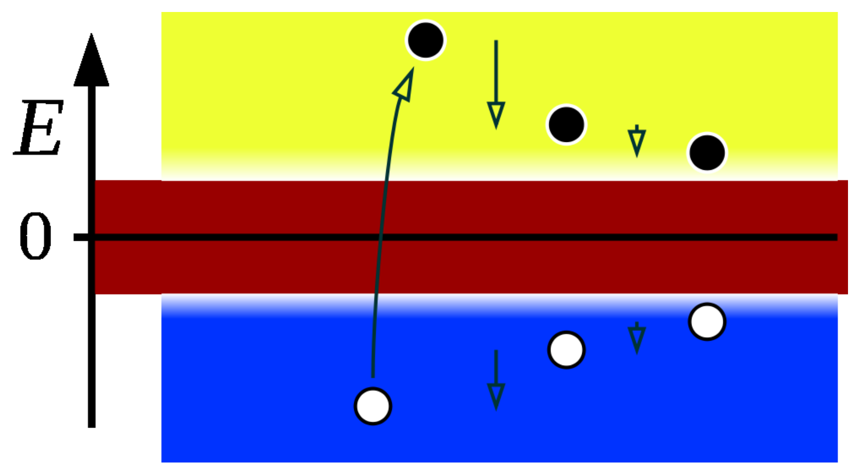
[10] Actually it cannot exceed 2, when we take into account two possible spin states

Dirac sea

- This fully-occupied state was interpreted as the vacuum
- Such vacuum is stable
- When we add an electron to the vacuum, it increases overall energy by $E > 0$. The dynamics of this additional particle is described by the positive-energy solutions of the Dirac equation.
- We may also **remove** one particle from the vacuum. The resulting state is called “hole”, and behaves like the absence of electron with $E = -\sqrt{\boldsymbol{p}^2 + m^2}$.

Therefore, the hole has[12]

- **positive** charge $e_{\text{hole}} = +|e|$
- **positive** energy $E_{hole} = +\sqrt{\boldsymbol{p}^2 + m^2}$
- opposite momentum $\boldsymbol{p}_{\text{hole}} = -\boldsymbol{p}$



[12]The analogy is an air bubble in water, compared to the drop of water in air: effectively, the bubble behaves like a drop with negative density.

Holes

The removal of fermion with negative energy **increases** the energy of the system. The vacuum has the lowest energy

Prediction of antiparticles

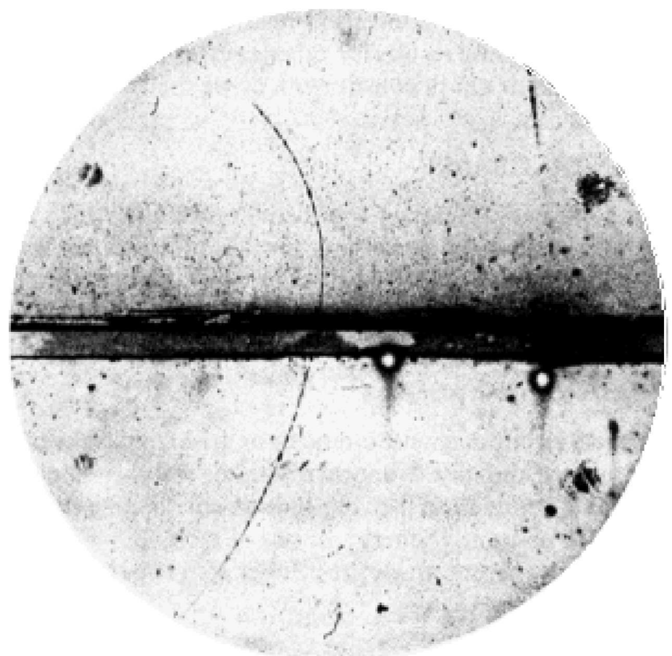
- The hole state is called the **antiparticle**
- For electron, there should exist a positively-charged antiparticle – **positron**. It was first predicted by Dirac.
- In 1932, positron was discovered by Anderson in cosmic rays
  Phys. Rev. 43, 491494 (1933) "*The positive electron*"
  http://link.aps.org/doi/10.1103/PhysRev.43.491

Prediction of antiparticles

From **Phys. Rev. 43, 491494 (1933) "*The positive electron*"**

http://link.aps.org/doi/10.1103/PhysRev.43.491

Abstract. Out of a group of 1300 photographs of cosmic-ray tracks in a vertical Wilson chamber 15 tracks were of positive particles which could not have a mass as great as that of the proton. From an examination of the energy-loss and ionization produced it is concluded that the charge is less than twice, and is probably exactly equal to, that of the proton. If these particles carry unit positive charge the curvatures and ionizations produced require the mass to be less than twenty times the electron mass. These particles will be called positrons. Because they occur in groups associated with other tracks it is concluded that they must be secondary particles ejected from atomic nuclei.



Other consequences of the Dirac sea

- Presence of the negative-energy levels means that you can create particle-antiparticle pairs out of “nowhere”

- Particles in the pair can be real, but they can be also virtual (i.e. $E^2 - \boldsymbol{p}^2 \neq m^2$)

- According to the Heisenberg uncertainty relation $\Delta E\ \Delta t\ \gtrsim\ 1$, if one measures the state of system two times, separated by a short period $\Delta t \ll 1/m$, one will find a state with $1, 2, 3, \ldots$ additional pairs.

- It means that we no longer work with definite number of particles: number of particles may change! (Contrary to non-relativistic quantum mechanics)

- We need an approach that naturally takes into account states with different number of particles (we will return to this point in this Lecture)

Interaction of light with the Dirac sea

Since vacuum is not "empty", electromagnetic waves act on it non-trivially:

- the virtual particle-antiparticle pairs are excited
- the pairs are polarized by the electric field of the wave
- polarization changes the propapagation of the wave (**vacuum polarization**)

Two different electromagnetic waves can act on each other, through the interaction of the polarized virtual pairs. Light can scatter off light even in the vacuum!

See V. Dunne 1202.1557

The vacuum behaves like a medium.