Topics in Standard Model

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Course material

Books used in this course:

Bjorken-Drell: James D. Bjorken, Sidney David Drell *Relativistic quantum mechanics*, Vol. 1, McGraw-Hill, 1964

Aitchison-Hey: I.J.R. Aitchison, A.J.G. Hey, *Gauge Theories in Particle Physics, Vol. 1: From Relativistic Quantum Mechanics to QED*, 3rd edition

Cheng-Li: T.-P. Cheng and L.-F. Li, *Gauge theory of elementary particle physics*. Clarendon Press, Oxford, 1982.

Landau-Lifshitz-vol2: L.D. Landau, E.M. Lifshitz (1975). The Classical Theory of Fields. Vol. 2 (4th ed.). Butterworth-Heinemann.

Landau-Lifshitz-vol3: L.D. Landau, E.M. Lifshitz (1977). Quantum

Mechanics: Non-Relativistic Theory. Vol. 3 (3rd ed.). Pergamon Press.

Landau-Lifshitz-vol4: V.B. Berestetskii, E.M. Lifshitz, L.P. Pitaevskii (1982). *Quantum Electrodynamics*. Vol. 4 (2nd ed.). Butterworth-Heinemann

Maxwell's equations

- In 1860s Maxwell wrote a set of field equations describing electromagnetism

$$\operatorname{div}\vec{E} = \rho \tag{1}$$

$$\operatorname{div}\vec{B} = 0 \tag{2}$$

$$\operatorname{curl}\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \tag{3}$$

$$\operatorname{curl}\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} + \vec{\jmath} \tag{4}$$

we will always work in vacuum, so that $\epsilon = \mu = 1$ and as a result $\vec{D} = \vec{E}$ and $\vec{B} = \vec{H}$! The CGS system of units is used, so that $\epsilon_0 = \mu_0 = 1$ and dimensions of $\vec{E}$ and $\vec{B}$ are the same!

- **Check** that as a consequence of Maxwell's equations (1)–(4)

1. One can introduce functions A_0 and $\vec{A}$ such that

$$\begin{aligned} \vec{B} &= \mathrm{curl}\,\vec{A} \\ \vec{E} &= -\frac{1}{c}\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}A_0 \end{aligned} \tag{5}$$

   and Eqs. (2)–(3) are satisfied **identically**

2. Eqs. (5) define A_0 and $\vec{A}$ up to an arbitrary scalar function $\lambda(t,\vec{x})$ such that changing

$$\begin{aligned} \vec{A} &\to \vec{A} + \vec{\nabla}\lambda \\ A_0 &\to A_0 - \frac{1}{c}\frac{\partial\lambda}{\partial t} \end{aligned} \tag{6}$$

   does not change the l.h.s. of Eq. (5)

3. Derive equations on $\vec{A}$ and $\vec{A}_0$ based on Eqs. (1) and (4)
4. Show that current should be conserved $\frac{\partial\rho}{\partial t} + \mathrm{div}\,\vec{\jmath} = 0$
5. Show that in a space without sources $\vec{E}$ and $\vec{B}$ obey wave equation:

$$\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} - \Delta\vec{E} = 0 \tag{7}$$

Maxwell's equations

- Maxwell has predicted that in the **empty space** electric and magnetic fields can exist – electromagnetic waves.

- Equation (7) means that the electromagnetic wave propagates through the space with the speed of light c.

- Hertz had discovered radio-waves and confirmed this prediction

- Maxwell's discovery meant that there is **no** "action-at-a-distance" but rather two electric (magnetic) bodies interact via emitting and receiving electromagnetic waves

- Maxwell's equations possess **Lorentz symmetry** .[1] To see this, define 4-vector $A_\mu = (A_0, \vec{A})$ (based on (5)) and define the field strength

$$F_{\mu\nu} \equiv \frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu} \tag{8}$$

[1] To remind yourself about 4-vectors, Lorentz transformations and Lorentz covariant formulation of electrodynamics, see Landau & Lifshitz, vol. 2, §§4–7, §§16–17, §23

where $x^\mu = (ct, \vec{x})$.

- **Find** the relation between $F_{\mu\nu}$ and $\vec{E}, \vec{B}$ so that (8) is equivalent to (5)

- The Maxwell's equations take the form

$$\frac{\partial F^{\mu\nu}}{\partial x^\mu} = j^\nu \tag{9}$$

where $j^\mu = (\rho, \vec{j})$ – 4-vector of electromagnetic current and $F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta}$ where $\eta_{\alpha\beta}$ is the Minkowski metric

- Lorentz symmetry is a rotation in the 4-dimensional space (given by the 4×4 antisymmetric matrix $\Lambda^\mu{}_\nu$) so that **any vector** translates as

$$\tilde{X}^\mu = \Lambda^\mu{}_\nu X^\nu \tag{10}$$

and field strength $F^{\mu\nu}$ translates as

$$\tilde{F}^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta F^{\alpha\beta} \tag{11}$$

Maxwell's equations

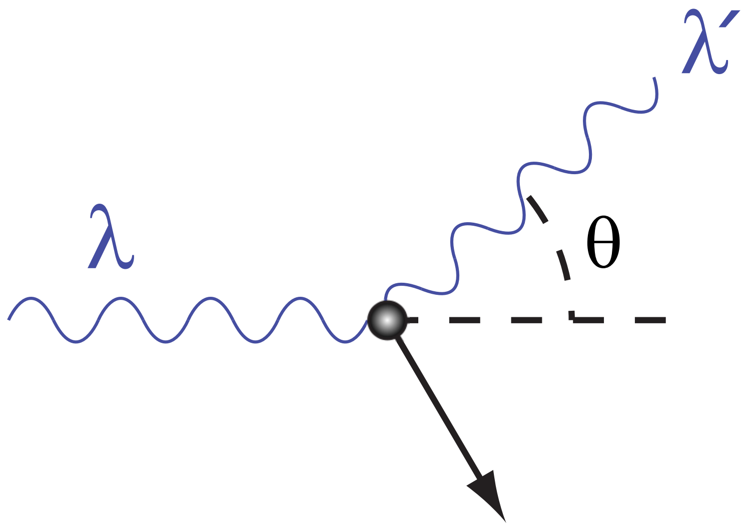
- **Show** that (9) is equivalent to (1) and (4)

- **Show** that Eq. (9) with $j^\mu = 0$ can be rewritten as 4 wave equations for 4 components of A_μ, supplemented by the Lorentz gauge condition $\partial_\mu A^\mu = 0$

Photons

- To explain some experiments (black body radiation; photoelectric effect) one was led to assume that the electromagnetic radiation is quantized and that a "quantum of light" – photon – exists (**Einstein 1905**)

- This has been confirmed by Compton in 1923 (**Compton scattering**)



$$\lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\theta)$$

  – X-rays, scattering of atomic electrons change wave length

  – Naturally explained by the elastic scattering of two particles (electron and photon)

  – Wave would not change its frequency (Thomson scattering)

- The atomic physics led to the creation of **Quantum mechanics**
- The quantum system is described by Schrödinger equation:

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = \hat{\mathcal{H}}\psi(x,t) \tag{12}$$

  where the operator $\hat{\mathcal{H}}$ is called **Hamiltonian**

- The operator Hamiltonian is built based on **correspondence principle**: **if** a classical system is described by the Hamiltonian $H(x,p)$ then the quantum system is described by $H(x,-i\hbar\partial_x)$. For example for a particle of mass m in external field we get

$$\text{Classical } H(x,p) = \frac{p^2}{2m} + V(x) \Rightarrow \text{Quantum } \hat{\mathcal{H}} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) \tag{13}$$

Solution of Schödinger equation

- Recall some important facts about the Schrödinger equation:[2]
  - Consider

$$0 = \int d^3x \left[\psi^* \left(i\hbar\frac{\partial \psi}{\partial t} - H\psi\right) - \psi\left(-i\hbar\frac{\partial \psi^*}{\partial t} - H\psi^*\right)\right]$$
$$\Leftrightarrow \frac{\partial}{\partial t}\int d^3x |\psi(x)|^2 = 0 \tag{14}$$

  - $|\psi(x)|^2$ is interpreted as probability density and $\int d^3x|\psi(x)|^2 = \mathrm{const}$ as the full probability
  - $\vec{J} = \frac{i\hbar}{2m}(\psi^*\nabla\psi - \psi\nabla\psi^*)$ is interpreted as a **probability density current** so that

$$\frac{\partial|\psi|^2}{\partial t} + \mathrm{div}\,\vec{J} = 0 \tag{15}$$

    Check this relation
  - Uncertainty principle $\Delta x \Delta p \sim \hbar$

[2] To remind yourself about this material, see e.g. Landau & Lifshitz, Vol. 3, §19

Solution of Schödinger equation

– Solution of the Schödinger equation for the hydrogen atom was predicting the energy spectrum

$$E_n = -\frac{m_e e^4}{2\hbar^2 n^2}, \quad n = 1, 2, \ldots \tag{16}$$

for each n the degeneracy (the number of states) of the energy level is $\sum_{l=0}^{n-1}(2l+1) = n^2$.

This latter fact contradicted to the observations as the lowest energy level of Hydrogen contained two electrons

Pauli Hamiltonian

- In order to explain the emission spectra of elements Pauli introduced a spin (*"a two-valued quantum degree of freedom"*). Its interpretation was not known. Pauli postulated **Pauli (or Pauli-Schrödinger) equation**

$$\hat{\mathcal{H}}_{\text{Pauli}} = \frac{1}{2m}\Big[\vec{\sigma}\cdot\big(-i\hbar\vec{\nabla} - e\vec{A}\big)\Big]^2 + eA_0\mathbb{1} \tag{17}$$

where **Pauli matrices** $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are defined as

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{18}$$

and $\mathbb{1}$ is 2×2 identity matrix. Pauli Hamiltonian acts on the **2-component wave-function**

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix} = \hat{\mathcal{H}}_{\text{Pauli}}\begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix} \tag{19}$$

Pauli matrices

Show that

- Pauli matrices are **Hermitian**
- For each Pauli matrix $\sigma_i^2 = \mathbb{1}$
- For any two Pauli matrices

$$\{\sigma_i, \sigma_j\} \equiv \sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij}\mathbb{1} \quad i, j = 1, 2, 3 \tag{20}$$

- $\sigma_x \sigma_y = i\sigma_z$ and in general

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \tag{21}$$

- As a consequence of (20) and (21) one finds

$$(\vec{\sigma} \cdot \vec{X})(\vec{\sigma} \cdot \vec{Y}) = \vec{X} \cdot \vec{Y}\mathbb{1} + i\vec{\sigma} \cdot (\vec{X} \times \vec{Y}) \tag{22}$$

- Use the identity (22) to rewrite the Pauli Hamiltonian as

$$\mathcal{H}_{\text{Pauli}} = \frac{\mathbb{1}}{2m}(-i\hbar\vec{\nabla} - e\vec{A})^2 - \frac{\hbar e}{2m}\vec{\sigma} \cdot \vec{B} + eA_0\mathbb{1} \tag{23}$$

Pauli matrices

– Compute the matrix $U = e^{\frac{i}{2}\alpha\sigma_z}$ (as the Taylor expansion of the exponent) and show that matrix U is **unitary**. Try to repeat this for arbitrary $e^{\frac{i}{2}\vec{\alpha}\cdot\vec{\sigma}}$

– Show that **any** 2×2 unitary matrix U with $\det U = 1$ can be represented as

$$U = \exp\Big(\frac{i}{2}(n_0 + \vec{n} \cdot \sigma)\Big); \qquad n_0^2 + \vec{n}^2 = 1 \tag{24}$$

Quantum mechanics and relativity

Attempts to blend quantum mechanics and special relativity led to the problem of infinities.[3]

- Electromagnetic energy of uniformly charged ball of radius r_e should be smaller than the total mass of electron ($m_e c^2$)?

$$m_e c^2 > \frac{e^2}{r_e} \Longrightarrow r_e > \frac{e^2}{mc^2} = 3 \times 10^{-13}\,\mathrm{cm} \quad (25)$$

  – larger than the size of atomic nucleus (Hydrogen $r_H \sim 1.75 \times 10^{-13}\,\mathrm{m}$)

- Pauli's idea of spin meant that electron possesses magnetic moment: $\mu_e = \frac{e\hbar}{2m_e c}$. The energy of magnetic sphere with radius r_e would be $\frac{\mu_e^2}{r_e^3}$ which would exceed $m_e c^2$ for $r \sim \frac{1}{m_e}\left(\frac{e\hbar}{c^4}\right)^{2/3}$ (even larger distances than (25!)

[3]See an interesting exposition of the historical perspectives at http://people.bu.edu/gorelik/cGh_Bronstein_UFN-200510_Engl.htm, **Sec. 4**

- Free Schrödinger equation is **not Lorentz invariant** (indeed, it is obtained from $E = \frac{p^2}{2m}$ dispersion relation)

- Replace r.h.s. by relativistic dispersion relation $E = \sqrt{p^2c^2 + m^2c^4}$

Relativistic Schrödinger equation ? $$i\hbar\frac{\partial\psi}{\partial t} = \boxed{\sqrt{-\hbar^2\vec{\nabla}^2 + m^2c^4}}\,\psi \quad (26)$$

- How to make sense of square root? Try to take square of Eq. (26)?

$$-\hbar^2\frac{\partial^2\psi}{\partial t^2} = \left(-\hbar^2c^2\nabla^2 + m^2c^4\right)\psi \Leftrightarrow \left(\Box + \left(\frac{mc}{\hbar}\right)^2\right)\psi = 0 \quad (27)$$

This is **Klein-Gordon equation**. Here $\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$ is the **D'Alembert operator**. **Show** that $\Box$ is Lorentz invariant

[3] The presentation of this topic follows Bjorken & Drell, Chap. 1, Sec. 1.1–1.3

- By taking square of (26) we have included **negative energy states** (with $E = -\sqrt{p^2c^2 + m^2c^4}$)

- Probabilistic interpretation is gone? Indeed, let us check **probability density** $|\psi|^2$:

$$\frac{\partial}{\partial t}\int d^3x\, |\psi(x)|^2 \neq 0 \tag{28}$$

- Repeating trick similar to Eq. (14) we find that the following current is conserved as a consequence of Klein-Gordon equation:

$$J^\mu = \psi^* \frac{\partial \psi}{\partial x^\mu} - \psi \frac{\partial \psi^*}{\partial x^\mu} \tag{29}$$

that is

$$\frac{1}{c}\frac{\partial J^0}{\partial t} + \operatorname{div}\vec{J} = 0 \tag{30}$$

Problems with Klein-Gordon equation

- Can we use J^0 as the probability density? **No!**

$$J^0 = \psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} = |\psi|^2 \frac{\partial(\arg \psi)}{\partial t} < 0 \tag{31}$$

for any **negative energy solution**

- Dirac (1928) had suggested that one needs linear in time evolution. Indeed, **if** $i\hbar\partial_t\psi = H\psi$ **than** the probability density of the form (28) is conserved **for any Hermitian Hamiltonian** (c.f. (14)).

- Dirac proposed the form

$$i\hbar\frac{\partial\psi}{\partial t} = \underbrace{\left[-i\hbar c\left(\alpha_x\frac{\partial}{\partial x} + \alpha_y\frac{\partial}{\partial y} + \alpha_z\frac{\partial}{\partial z}\right) + \beta mc^2\right]}_{\text{Dirac Hamiltonian}}\psi \tag{32}$$

- Take square of Dirac Hamiltonian. We should arrive to the Klein-Gordon equation. This means that for $i, j = 1, 2, 3$

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij} \tag{33}$$

$$\alpha_i\beta + \beta\alpha_i = 0 \tag{34}$$

[4] The presentation of this topic similar to that of Bjorken & Drell, Chap. 1, Sec. 1.1–1.3

and additionally $\beta^2 = 1$

- $\Rightarrow$Both α_i nor β are **not ordinary numbers** (they do not commute).
- Eq. (33) looks very much like Eq. (20). But guess that $\alpha_i = \sigma_i$ is **wrong** there are only 3 Pauli matrices and we need **4**
- Another guess:[6]

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}; \qquad \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \tag{35}$$

Show that properties (33)–(34) are satisfied. **Find** different representation of α_i, β

[6]**Show** that α_i, β are **even-dimensional matrices**

Properties of Dirac gamma-matrices [8]

- Define 4 γ-matrices (or **Dirac gamma-matrices**):

$$\gamma^0 = \beta; \quad \gamma^i = \beta\alpha^i \tag{36}$$

- Define 4-vector $\gamma^\mu = (\gamma^0, \gamma^1, \gamma^2, \gamma^3)$
- Dirac equation can be rewritten as

$$i\hbar\gamma^\mu \frac{\partial\psi}{\partial x^\mu} = 0 \tag{37}$$

  – explicitly Lorentz-covariant

- Properties of γ-matrices:

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbb{1} \tag{38}$$

[7] See Bjorken & Drell, Chap. 2, Sec. 2.1–2.2

Properties of Dirac gamma-matrices [9]

- Define
$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \tag{39}$$

- Show that $[\sigma^{\mu\nu}, \sigma^{\lambda\rho}]$ commute as operators of Lorentz rotation in $3+1$

Standard Model: great success of particle physics

Standard Model of

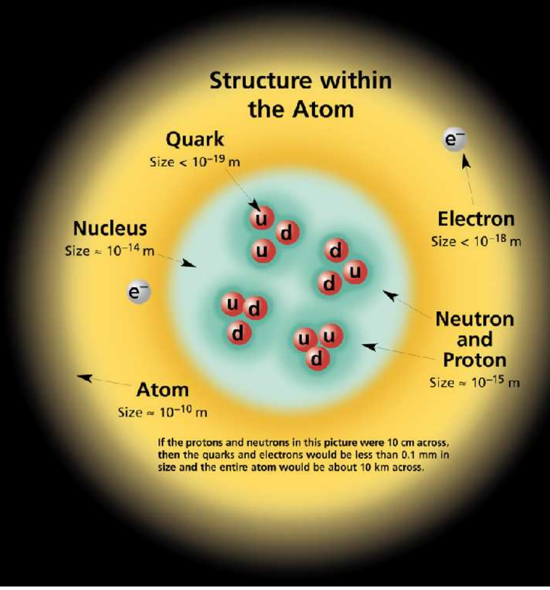
FUNDAMENTAL PARTICLES AND INTERACTIONS

FERMIONS

matter constituents
spin = 1/2, 3/2, 5/2, ...

Leptons spin = 1/2		
Flavor	Mass GeV/c^2	Electric charge
ν_e electron neutrino	$<1\times10^{-8}$	0
e electron	0.000511	−1
ν_μ muon neutrino	<0.0002	0
μ muon	0.106	−1
ν_τ tau neutrino	<0.02	0
τ tau	1.7771	−1

Quarks spin = 1/2		
Flavor	Approx. Mass GeV/c^2	Electric charge
u up	0.003	2/3
d down	0.006	−1/3
c charm	1.3	2/3
s strange	0.1	−1/3
t top	175	2/3
b bottom	4.3	−1/3



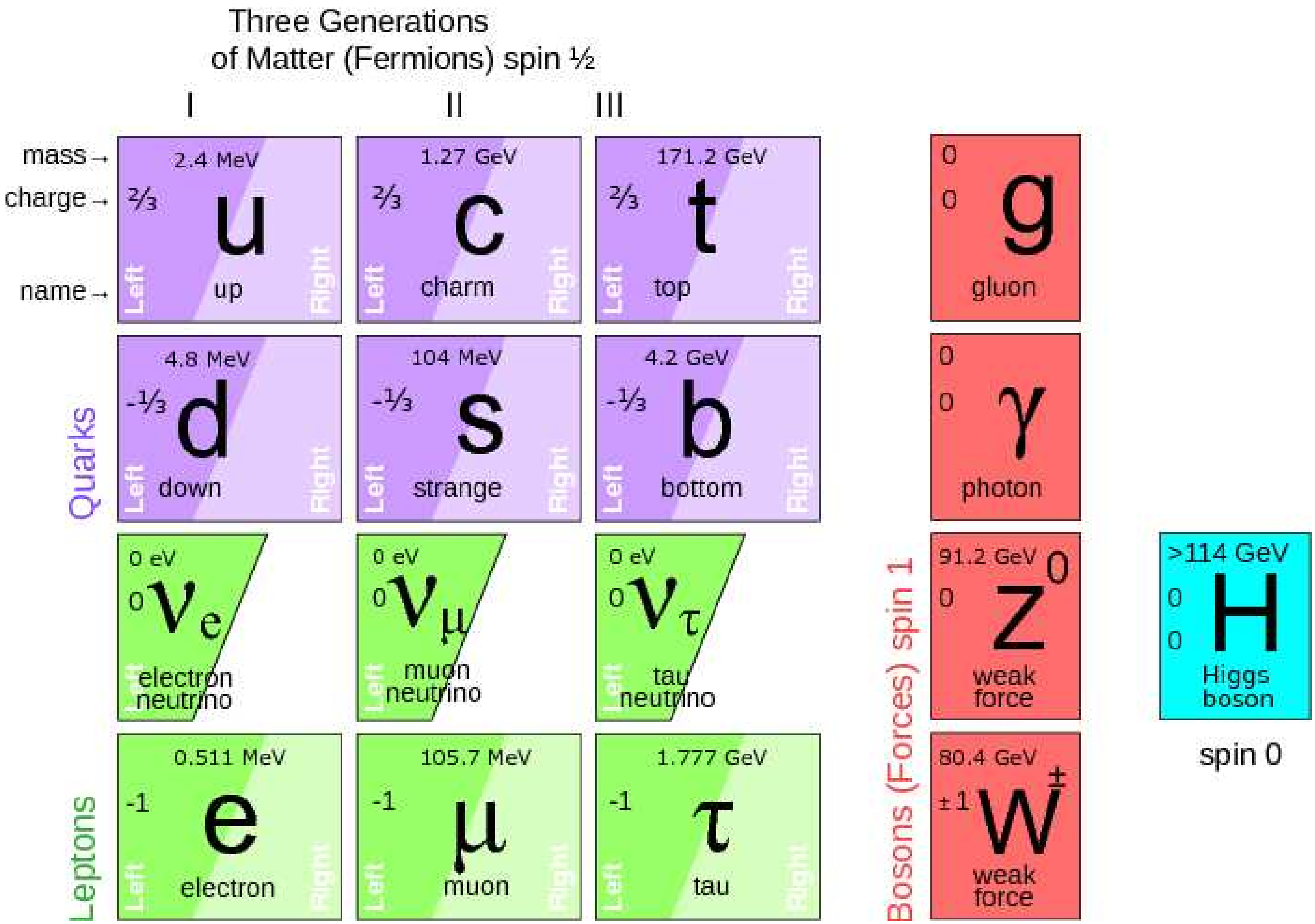
BOSONS

force carriers
spin = 0, 1, 2, ...

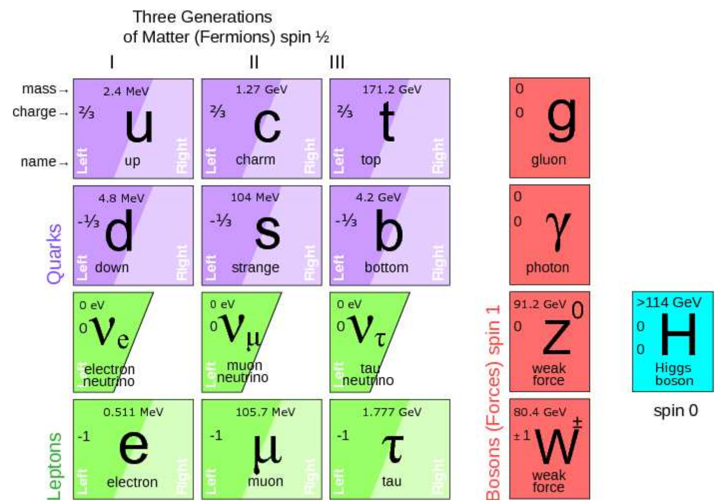
Unified Electroweak spin = 1		
Name	Mass GeV/c^2	Electric charge
γ photon	0	0
W^-	80.4	−1
W^+	80.4	+1
Z^0	91.187	0

Strong (color) spin = 1		
Name	Mass GeV/c^2	Electric charge
g gluon	0	0

Standard Model



Standard Model



- The Standard Model describes all the confirmed data obtained using particle accelerators and has enabled many successful theoretical predictions.

- It has been tested with amazing accuracy, and its calculable quantum corrections play an essential role. Its only missing feature is a particle, called the Higgs boson, whose coupling to the other particles is believed to generate their masses.

Standard Model and particle astrophysics

The SM has also been successful in explaining phenomena in the Early Universe and certain astrophysical systems.

- It plays an essential role in the epoch of Big Bang nucleosynthesis – observed abundances of elements are sensitive to details of the Standard Model.
- Energy production of main-sequence stars is controlled by nuclear physics and weak β decay.
- Weak neutral-current processes are important for neutrino emission from the collapsing core and ejection of outer layers of the supernova progenitor.
- Propagation of cosmic rays through the Universe.

End of Lecture 1

End of Lecture I