Particle Physics of the early Universe

Alexey Boyarsky

Spring semester 2015

Why do you need that?

Plan of today's lecture

- Let us apply the formalism that we developed in the previous Lecture to the **electrodynamics** – a system of interacting fermions plus dynamical electromagnetic field

- We will consider three different cases:

  – Electron is scattering in the external electromagnetic field
  – Electron is scattering on the (dynamical) proton
  – Electron is scattering on its own anti-particle (positron)

Electron scattering in Coulomb field

- In non-relativistic quantum mechanics if Hamiltonian has the form $\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \hat{V}$ then the probability of transition between an initial state $\psi_i(x)$ and the final state $\psi_f(x)$ of **unperturbed** Hamiltonian $\hat{\mathcal{H}}_0$ is given by (Landau & Lifshitz, vol. 3, § 43):

$$dw_{if} = \frac{2\pi}{\hbar}|V_{if}|^2\delta(E_i - E_f)d\nu_f \tag{1}$$

where $|V_{if}|$ is the matrix element between initial and final states; and dn_f is the number of final states with the energy E_f (degeneracy of the energy level).

- The interaction that perturbs the Hamiltonian is given by

$$V_{\text{int}} = e\gamma^0\gamma^\mu A_\mu(x) \tag{2}$$

recall that electric current $j^\mu = e\bar{\psi}(x)\gamma^\mu\psi(x)$

[0]Following Bjoren & Drell, Sec. 7.1

Electron scattering in Coulomb field

- If we consider static point source with the Coulomb field

$$A_0(\boldsymbol{x}) = \frac{Ze}{4\pi|\boldsymbol{x}|}, \quad \vec{A} = 0 \tag{3}$$

and wave-functions[1]

$$\psi_i(x) = C_1 u_s(p_i) e^{ip_i \cdot x} \quad , \quad \psi_f(x) = C_2 u_r(p_f) e^{ip_f \cdot x} \tag{4}$$

Find C_1 that ψ_i in (4) has the correct normalization per unit flux in the relativistic case. Normalization of u is given by (8)

- Using (2–4) we write the matrix element

$$V_{if} = C_1 C_2 \bar{u}_r(p_f) \gamma^0 u_s(p_f) \int d^3\boldsymbol{x} e^{i\boldsymbol{x}\cdot(\boldsymbol{p}_i - \boldsymbol{p}_f)} A_0(\boldsymbol{x}) \tag{5}$$

[1] Here u_s, $\bar{u}_r$ are 4-component spinors – solution of the Dirac equations $(\gamma \cdot p - m)u_s = 0$, $\bar{u}_r(\gamma \cdot p + m) = 0$, $s = \pm$, $r = \pm$ – polarizations of spin.

Electron scattering in Coulomb field

- Degeneracy of a final state with E_f is given by

$$d\nu_f = \mathbf{2} \times \underbrace{\frac{d^3\boldsymbol{p}_f}{(2\pi)^3} \int\limits_{p_0>0} dp_0\, \delta(p^2 - m^2)}_{\text{density of states}} = \frac{d^3\boldsymbol{p}_f}{(2\pi)^3 E_f} \tag{6}$$

- Eq. (6) and normalization of the final state ψ_f should agree with each other in such a way that

$$\int d\nu_f\, \bar{\psi}_f(x)\psi_f(x') = \delta^{(3)}(x - x') \tag{7}$$

  Notice that this condition fixes the normalization of the spinor u_f in Eq. (4).

- $u_s(p_i)$ and $u_r(p_f)$ carry the information about spin-polarizations of initial and final states. We can sum over these states (i.e. the experiment does not measure the polarizations). This can be done using the identity

$$\sum_s u_s(p)\bar{u}_s(p) = \left(\frac{\not{p}+m}{2m}\right) \tag{8}$$

(take into account $(\not{p}-m)(\not{p}+m) = p^2 - m^2$)

- As a result we get

$$\begin{aligned} dw_{if} &= 2\pi|V_{if}|^2\delta(E_i - E_f)\frac{d^3\boldsymbol{p}_f}{(2\pi)^3 E_f} \\ &= Z^2\,(4\pi\alpha)^2|C_1|^2|C_2|^2\frac{|\bar{u}_r(p_f)\gamma^0 u_s(p_i)|^2}{|\boldsymbol{p}_i - \boldsymbol{p}_f|^4}\frac{d^3\boldsymbol{p}_f}{(2\pi)^3 E_f}\delta(E_i - E_f) \end{aligned} \tag{9}$$

Electron scattering in Coulomb field

- as a result the sum over initial and final states in (9) becomes

$$\sum_{r,s} |\bar{u}_r(p_i)\gamma^0 u_s(p_f)|^2 = \sum_{r,s} \bar{u}_r\gamma^0 u_s\bar{u}_s\gamma^0 u_r = \mathrm{Tr}\left(\gamma^0\frac{\not{p}_i + m}{2m}\gamma^0\frac{\not{p}_f + m}{2m}\right)$$

- the spin sum rule leads to

averaging over initial polarizations

$$\frac{1}{2}\sum_s\sum_r |\bar{u}_r\gamma^0 u_s|^2 \equiv F(p_i, p_f, m) = (E_i E_f + \boldsymbol{p}_i\boldsymbol{p}_f + m^2)/2m^2 \qquad (10)$$

summing over the final polarizations

- Using () and representing $d^3\boldsymbol{p}_f = d\Omega\,\boldsymbol{p}_f^2 d\boldsymbol{p}_f$ we find the differential cross-section

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2 m E_i}{4|\boldsymbol{p}_i|^4\sin^4(\frac{\theta}{2})}F(p_i, p_f, m) \qquad (11)$$

– which coincides with the familiar Rutherford scattering (up to the function $F(\dots)$ that depends on the spins/polarizations of the particles

Electron scattering on proton

- Consider next the situation when the electromagnetic field is created by other particle ("**proton**")

- While the formulas (4)–(6) remain true, the expression for A_μ changes

- Notice that in general $A_\mu(x)$ created by the moving proton will be **time-dependent** so instead of Eq. (1) the formula

$$dw_{if} = \frac{2\pi}{\hbar}|V_{if}|^2\delta(E_i - E_f - q_0)dn_f \tag{12}$$

where q_0 is the frequency of the perturbation $A_\mu(t, \boldsymbol{x}) \propto e^{-iq_0 t}$

- If proton is described by a spinor

$$\Psi = U(P)e^{-iP\cdot x}, \qquad U(P)\text{–4 component spinor} \tag{13}$$

then its electric current is

$$J^\mu(y) = \bar{\Psi}_f(y)\gamma^\mu\Psi_i(y) \quad (14)$$

(the form of Ψ_i and $\bar{\Psi}_f$ is the same as Eq. (4) with $m \to M_p$ and different momenta)

- The electromagnetic field obeys the Klein-Gordon equation

$$\Box A_\mu = J_\mu \quad \text{or} \quad A_\mu(x) = \frac{1}{\Box} J_\mu(y) \quad (15)$$

- Very naively, the operator $\Box^{-1}$ (inverse to the Klein-Gordon operator) can be easily constructed if one considers (15) in Fourier space:[2]

$$p^2 \tilde{A}_\mu(q) = \tilde{J}_\mu(q) \quad \text{or} \quad \tilde{A}_\mu(q) = \frac{1}{q^2}\tilde{J}_\mu(q) \quad (16)$$

[2] We denote by $\tilde{A}_\mu(p)$ and $\tilde{J}_\mu(p)$ Fourier transform

- where

$$\tilde{J}_\mu(q) \equiv \int d^4x\, e^{-iq\cdot x}\bar{\Psi}_f(x)\gamma^\mu\Psi_i(x) \quad (17)$$

$$= \sqrt{\frac{M_p^2}{E_i^{(p)}E_f^{(p)}}}\bar{U}_r(P_i)\gamma_\mu U_s(P_f)\int d^4x\, e^{-i(q-P_f-P_f)\cdot x} \quad (18)$$

$$= \sqrt{\frac{M_p^2}{E_i^{(p)}E_f^{(p)}}}\bar{U}_r(P_i)\gamma_\mu U_s(P_f)\delta^{(4)}(q-P_f-P_f) \quad (19)$$

- Notice that in Eq. (5) we only need $\tilde{A}_\mu(p_i - p_f)$. The resulting expression is then equivalent to (5) if one substitutes

$$\gamma^0\frac{Z}{|\boldsymbol{q}|^2} \to \gamma^\mu\frac{1}{q^2}\sqrt{\frac{M_p^2}{E_i^{(p)}E_f^{(p)}}}\bar{U}_r(P_i)\gamma_\mu U_s(P_f)\delta^{(4)}(q-P_f-P_f) \quad (20)$$

where $q = p_f - p_i$ – transferred 4-momentum and $\boldsymbol{q}$ is its spatial component. 4-spinors U_s and $\bar{U}_r$ are the in- and out- 4-spinors of a proton.

Electron scattering on proton

- From Eq. (5) and (20) we find

$$V_{if} = \int d^3\boldsymbol{x} \int d^3\boldsymbol{q}\, (\bar{u}\gamma^\mu u) e^{-i(q-p_i+p_f)\cdot x} \tilde{A}_\mu(q) \tag{21}$$

where $\tilde{A}_\mu(q)$ is given by r.h.s. of (20):

- The resulting V_{if} is proportional to

$$V_{if} \propto \delta^3(\boldsymbol{p}_i - \boldsymbol{p}_f + \boldsymbol{P}_i - \boldsymbol{P}_f) \tag{22}$$

- That is the result looks like a scattering of electron in external field (9) where the external field A_μ is the field created by the proton (14).

- As a result the probability dw_{if} has the form very similar to Eq. (9)

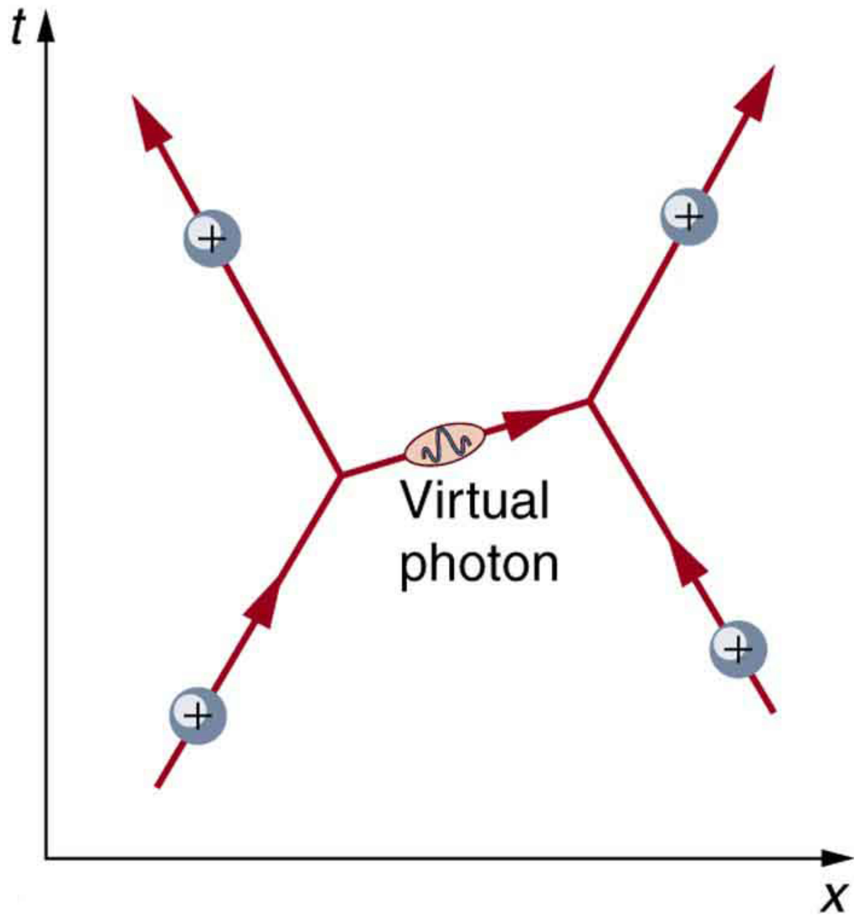
Electron scattering on proton

for $Z = 1$ and

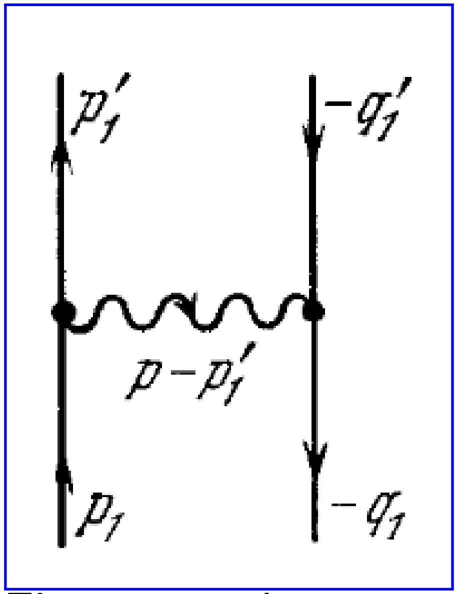
$$dw_{if} =(\dots)\frac{\left|\bar{u}_r(p_f)\gamma^\mu u_s(p_i)\right|^2\left|\bar{U}_{r'}(P_f)\gamma^\mu U_{s'}(P_i)\right|^2}{|\boldsymbol{p}_i-\boldsymbol{p}_f|^4} \times \delta(E_i+E_i^{(p)}-E_f-E_f^{(p)})\delta^3(\boldsymbol{p}_i-\boldsymbol{p}_f+\boldsymbol{P}_i-\boldsymbol{P}_f) \tag{23}$$

Interaction of light with the Dirac sea

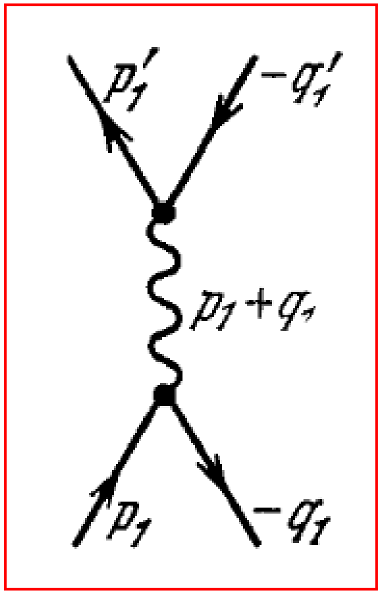
Interaction of charged particles goes via exchange of virtual photon quanta



Electron-positron scattering[3]



$+$



Electron-positron scattering

- Two differences between **positron** and **proton**. One is trivial – mass (not important for our computations so far).

- blue diagram is similar to what we have seen before (electron-proton scattering with proton $\to$ positron).

- Another difference: electron plus positron can **convert** into a **photon**

- red diagram : electron+positron disappear in the intermediate state and only a new particle (photon) stays

$\Rightarrow$need a quantum theory of photon

[2]See Bjoren & Drell, Sec. 7.9

Electromagnetic field as collection of oscillators [4]

- Consider the solution of wave equation for vector potential $\vec{A}$ (impose $A_0 = 0$ and $\mathrm{div}\,\vec{A} = 0$)

$$\frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \Delta \boldsymbol{A} = 0 \tag{24}$$

- Solution

$$\boldsymbol{A}(x,t) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\left[\boldsymbol{a_k}(t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \boldsymbol{a}^*_{\boldsymbol{k}}(t)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right] \tag{25}$$

where the complex functions $\boldsymbol{a_k}(t)$ have the following time dependence

$$\boldsymbol{a_k}(t) = \boldsymbol{a_k}e^{-i\omega_{\boldsymbol{k}}t}, \qquad \omega_{\boldsymbol{k}} = |\boldsymbol{k}| \tag{26}$$

[3] See Landau & Lifshitz, Vol. 4, §2

Generalized coordinate and momentum

- $\boldsymbol{a_k}(t)$ and $\boldsymbol{a^*_k}(t)$ obey the following equations:

$$\dot{\boldsymbol{a}}_{\boldsymbol{k}}(t) = -i\omega_{\boldsymbol{k}}\boldsymbol{a}_{\boldsymbol{k}}(t) \quad , \quad \dot{\boldsymbol{a}}^*_{\boldsymbol{k}}(t) = i\omega_{\boldsymbol{k}}\boldsymbol{a}^*_{\boldsymbol{k}}(t) \tag{27}$$

- Notice that we re-wrote partial differential equation (25) second order in time into a set of ordinary differential equations for infinite set of functions $\boldsymbol{a_k}(t)$ and $\boldsymbol{a^*_k}(t)$. **Any solution of the free Maxwell's equation** is parametrized by the (infinite) set of complex number $\{\boldsymbol{a_k}, \boldsymbol{a^*_k}\}$

- To make the meaning of Eqs. (30) clear, let us introduce their imaginary and real parts.

$$\left.\begin{aligned} \boldsymbol{Q_k} &\equiv \boldsymbol{a_k} + \boldsymbol{a^*_k} \\ \boldsymbol{P_k} &\equiv -i\omega_{\boldsymbol{k}}\big(\boldsymbol{a_k} - \boldsymbol{a^*_k}\big) \end{aligned}\right\} \overset{\text{dynamics}}{\Longrightarrow} \left\{\begin{aligned} \dot{\boldsymbol{Q}}_{\boldsymbol{k}} &= \boldsymbol{P_k} \\ \dot{\boldsymbol{P}}_{\boldsymbol{k}} &= -\omega^2_{\boldsymbol{k}}\boldsymbol{Q_k} \end{aligned}\right. \tag{28}$$

Hamiltonian of electromagnetic field

- Hamiltonian (total energy) of electromagnetic field is given by

$$\mathcal{H} = \frac{1}{2}\int d^3\boldsymbol{x}\left[\boldsymbol{E}^2 + \boldsymbol{B}^2\right] = \frac{1}{2}\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\left[\boldsymbol{E}_{\boldsymbol{k}}^2 + \boldsymbol{B}_{\boldsymbol{k}}^2\right] \qquad (29)$$

- Using mode expansion (25) and definition (28) we can write with frequencies $\omega_{\boldsymbol{k}}$

$$\mathcal{H}[\boldsymbol{Q}_{\boldsymbol{k}}, \boldsymbol{P}_{\boldsymbol{k}}] = \frac{1}{2}\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\left[\boldsymbol{P}_{\boldsymbol{k}}{}^2 + \omega_{\boldsymbol{k}}^2 \boldsymbol{Q}_{\boldsymbol{k}}{}^2\right] \qquad (30)$$

- Therefore dynamical equations (28) are nothing by the Hamiltonian equations

$$\dot{\boldsymbol{Q}}_{\boldsymbol{k}} = \frac{\partial \mathcal{H}}{\partial \boldsymbol{P}_{\boldsymbol{k}}} \quad , \quad \dot{\boldsymbol{P}}_{\boldsymbol{k}} = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{Q}_{\boldsymbol{k}}} \qquad (31)$$

with Hamiltonian (30)

Hamiltonian of electromagnetic field

- Eqs. (30)–(31) describe Hamiltonian dynamics of a sum of independent oscillators with frequencies $\omega_{\boldsymbol{k}}$

Classical electromagnetic field can be considered as an infinite sum of oscillators with frequencies $\omega_{\boldsymbol{k}}$

- **Recall:** for quantum mechanical oscillator, described by the Hamiltonian

$$\hat{\mathcal{H}}_{\text{osc}} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{m\omega^2}{2}x^2 \tag{32}$$

one can introduce **creation and annihilation operators**:

$$\hat{a}^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega x + \hbar\partial_x) \quad ; \quad \hat{a} = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega x - \hbar\partial_x) \tag{33}$$

- Commutation $[\hat{a}, \hat{a}^\dagger] = 1$
- Hamiltonian can be rewritten as $\hat{\mathcal{H}}_{\text{osc}} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$

Properties of creation/annihilation operators

- Commutation $[\hat{a}, \hat{a}^\dagger] = 1$

- If one defines a **vacuum** $|0\rangle$, such that $\hat{a}\,|0\rangle = 0$ (**Fock vacuum**) then a state $|n\rangle \equiv (\hat{a}^\dagger)^n\,|0\rangle$ is the eigenstate of the Hamiltonian (32) with $E_n = \hbar\omega(n + \frac{1}{2})$, $n = 0, 1, \ldots$

- Given $|n\rangle$, $n > 0$, $\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$ and $\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle$

- Time evolution of the operators $\hat{a}, \hat{a}^\dagger$:

$$i\hbar\frac{\partial\hat{a}}{\partial t} = [\mathcal{H}_{\text{osc}}, \hat{a}] \tag{34}$$

and Hermitian conjugated for $\hat{a}^\dagger$

Birth of quantum field theory

- Dirac (1927) proposes to treat radiation as a collection of **quantum** oscillators

Paul A.M. Dirac *Quantum theory of emission and absorption of radiation*

Proc.Roy.Soc.Lond. A114 (1927) 243

- Take the classical solution (25)

$$\boldsymbol{A}(x,t) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\left[\boldsymbol{a}_{\boldsymbol{k}}(t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \boldsymbol{a}^*_{\boldsymbol{k}}(t)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right]$$

- Introduce creation/annihilation operators $\hat{a}_{\boldsymbol{k}}$, $\hat{a}^\dagger_{\boldsymbol{k}}$

$$[\hat{a}_{\boldsymbol{k}}, \hat{a}^\dagger_{\boldsymbol{p}}] = \hbar\delta_{\boldsymbol{k},\boldsymbol{p}} \qquad [\hat{a}_{\boldsymbol{k}}, \hat{a}_{\boldsymbol{p}}] = 0 \tag{35}$$

Birth of quantum field theory

- Replace Eq. (25) with a quantum operator

$$\hat{\boldsymbol{A}}(x,t) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\left[\hat{\boldsymbol{a}}_{\boldsymbol{k}}(t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \hat{\boldsymbol{a}}^\dagger_{\boldsymbol{k}}(t)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right] \quad (36)$$

- Operator $\boxed{\hat{\boldsymbol{a}}^\dagger_{\boldsymbol{k}}}$ **creates photon with momentum k and frequency $\omega_{\boldsymbol{k}}$**

- Operator $\boxed{\hat{\boldsymbol{a}}_{\boldsymbol{k}}}$ **destroys photon with momentum k and frequency $\omega_{\boldsymbol{k}}$** (if exists in the initial state)

- State without photons $\leftrightarrow$ Fock vacuum:

$$\hat{\boldsymbol{a}}_{\boldsymbol{k}}\,|0\rangle = 0 \quad \forall \boldsymbol{k} \quad (37)$$

Birth of quantum field theory

- State with N photons with momenta $\boldsymbol{k}_1, \boldsymbol{k}_2, \ldots, \boldsymbol{k}_N$:

$$|\boldsymbol{k}_1, \boldsymbol{k}_2, \ldots, \boldsymbol{k}_N\rangle = \hat{\boldsymbol{a}}^\dagger_{\boldsymbol{k}_1} \hat{\boldsymbol{a}}^\dagger_{\boldsymbol{k}_2} \ldots \hat{\boldsymbol{a}}^\dagger_{\boldsymbol{k}_N} |0\rangle \tag{38}$$

- Let us find the average of the operator (36) between a vacuum and the **1-photon state** $|\boldsymbol{k}\rangle \equiv \hat{\boldsymbol{a}}^\dagger_{\boldsymbol{k}} |0\rangle$ (recall that we are working in the gauge $A_0 = 0$)

$$\langle 0| \hat{\boldsymbol{A}}(x) \Big(\hat{\boldsymbol{a}}^\dagger_{\boldsymbol{k}} |0\rangle \Big) = \frac{\boldsymbol{\epsilon}(\boldsymbol{k})}{\sqrt{2\omega_{\boldsymbol{k}}}} e^{-i\omega_{\boldsymbol{k}} t + i\boldsymbol{k}\cdot\boldsymbol{x}} \tag{39}$$

where the polarization 3-vectors $\boldsymbol{\epsilon}(\boldsymbol{k})$ are such that $\boldsymbol{\epsilon} \cdot \boldsymbol{k} = 0$ and $\boldsymbol{\epsilon}^2 = 1$

Quantum electrodynamics (QED) – first quantum field theory has been created. Free fields with interaction treated **perturbatively** in *fine-structure constant:* $\alpha = \frac{e^2}{\hbar c}$

Second order perturbation theory

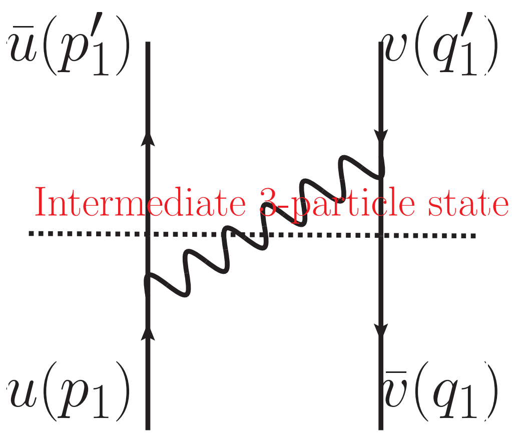
- The Dirac Hamiltonian $\mathcal{H}_0$ is perturbed by $\hat{V}$ given by
$$\hat{V} = \gamma^0\gamma^\mu \hat{A}_\mu(x)$$

- Let us start with the blue diagram

- We could have proceeded as in the case of electron-proton interaction – find the electromagnetic field, created by the positron and compute the scattering of electron in this field. Instead, we do this computation in a different way, and then demonstrate that the result is the same

- A full system of **intermediate states** $|n\rangle$ contains both electron, positron **and** photon. That is
  – the initial state $|i\rangle = |e^-(p_1), e^+(q_1)\rangle \otimes |0\rangle$
  – the final state $|f\rangle = |e^-(p_1'), e^+(q_1')\rangle \otimes |0\rangle$
  – and the intermediate state $|n\rangle = |e^-(p_1'), e^+(q_1)\rangle \otimes |\boldsymbol{k}\rangle$
  (momentum of electron in the intermediate state is equal to its final momentum!)

Computation of the matrix element



– the initial state $|i\rangle = |e^-(p_1), e^+(q_1)\rangle \otimes |0\rangle$.

The energy of initial state: $E_i = E(\boldsymbol{q}_1) + E(\boldsymbol{p}_1)$

– The intermediate state $|n\rangle = |e^-(p_1'), e^+(q_1)\rangle \otimes |\boldsymbol{k}\rangle$.

The energy of intermediate state: $E_n = E(\boldsymbol{q}_1) + E(\boldsymbol{p}_1') \pm \omega_k$

– The final state $|f\rangle = |e^-(p_1'), e^+(q_1')\rangle \otimes |0\rangle$.

- The matrix element for the process can be described as follows:

$$M_{if} = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \langle i| \hat{V} |n\rangle \frac{1}{E_i - E_n} \langle n| \hat{V} |f\rangle \tag{40}$$

- The matrix element $V_{in} \equiv \Big(\langle 0| \otimes \langle e^+e^-|\Big) \hat{V} \Big(|e^+e^-\rangle \otimes |\boldsymbol{k}\rangle\Big)$ is given by the following expression

Computation of the blue diagram

$$
\begin{aligned}
V_{in} =& \bar{u}(p_1')\gamma^\mu u(p_1)\frac{\epsilon_\mu}{\sqrt{2\omega_{\boldsymbol{k}}}}\int d^3\boldsymbol{x}\, e^{i(\boldsymbol{p}_1-\boldsymbol{p}_1'-\boldsymbol{k})\cdot\boldsymbol{x}} e^{i\left(E(\boldsymbol{p}_1)-\omega_{\boldsymbol{k}}-E(\boldsymbol{p}_1')\right)t} + \{\omega_{\boldsymbol{k}} \to -\omega_{\boldsymbol{k}}\} \\
=& \bar{u}(p_1')\gamma^\mu u(p_1)\frac{\epsilon_\mu}{\sqrt{2\omega_{\boldsymbol{k}}}}\delta^{(3)}(\boldsymbol{p}_1-\boldsymbol{p}_1'-\boldsymbol{k})e^{i\left(E(\boldsymbol{p}_1)-\omega_{\boldsymbol{k}}-E(\boldsymbol{p}_1')\right)t} + \{\omega_{\boldsymbol{k}} \to -\omega_{\boldsymbol{k}}\}
\end{aligned}
\tag{41}
$$

(the positron state is the same in $|i\rangle$ and $|n\rangle$ and therefore $\langle e^+|\,|e^+\rangle = 1$)

- Notice that the expression (41) contains explicit time dependence as $E(\boldsymbol{p}_1) \neq \omega_{\boldsymbol{k}} + E(\boldsymbol{p}_1')$. This is the indication of the fact that 3-momentum **and** energy cannot be conserved at the same time.
- Similarly $V_{nf} = \delta^{(3)}(\boldsymbol{q}_1 - \boldsymbol{q}_1' + \boldsymbol{k})\bar{v}(q_1)\gamma^\mu v(q_1')\frac{\epsilon_\mu}{\sqrt{2\omega_{\boldsymbol{k}}}}e^{i(E(\boldsymbol{q}_1)+\omega_{\boldsymbol{k}}-E(\boldsymbol{q}_1'))t}$
- As a result M_{if} is given by (after the integral over $dn \equiv \frac{d^3\boldsymbol{k}}{(2\pi)^3}$):

$$
M_{if} = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{V_{in}V_{nf}}{E_i - E_n} =
$$

Computation of the blue diagram

$$M_{if} = e^{i(E_i-E_f)t}\delta^{(3)}(\boldsymbol{p}_1' + \boldsymbol{q}_1' - \boldsymbol{p}_1 - \boldsymbol{q}_1)\frac{(\bar{v}(q_1)\gamma^\mu v(q_1'))\big(\bar{u}(p_1')\gamma^\mu u(p_1)\big)}{\big(E(\boldsymbol{p}_1) - E(\boldsymbol{p}_1')\big)^2 - \omega_{\boldsymbol{k}}^2}\bigg|_{\boldsymbol{k}=\boldsymbol{p}_1-\boldsymbol{p}_1'} \quad (42)$$

- **Show** that the denominator of (42) is equal to $\frac{1}{k^2}$ where k_μ is 4-vector equal to the difference $(p_1 - p_1')_\mu$.
- notice, that time dependence will cancel out from (42) when we take into account $\delta(E_i - E_f)$
- Blue diagram can be thought of as 2nd order perturbation theory with the 3-particle intermediate state $|n\rangle$, containing photon.
- Notice that the matrix element (42) coincides with the matrix element (22) (taking into account (21))

Computation of the red diagram

- There is another intermediate state where there is **only photon** — red diagram
- In this case we have different intermediate state: $|n\rangle = |0\rangle \otimes |\boldsymbol{k}\rangle$. The energy of intermediate state: $E_n = \pm\omega_k$
- Matrix element

$$V_{in} \equiv \langle e^+e^- \big| \hat{V} \, |\boldsymbol{k}\rangle = \bar{v}(q_1)\gamma^\mu u(p_1)\frac{\epsilon_\mu}{\sqrt{2\omega_{\boldsymbol{k}}}}\int d^3\boldsymbol{x}\, e^{i(\boldsymbol{p}_1+\boldsymbol{q}_1-\boldsymbol{k})\cdot\boldsymbol{x}}e^{i(E_i-\omega_{\boldsymbol{k}})t}$$
$$=\bar{v}(q_1)\gamma^\mu u(p_1)\frac{\epsilon_\mu}{\sqrt{2\omega_{\boldsymbol{k}}}}\delta^{(3)}(\boldsymbol{p}_1+\boldsymbol{q}_1-\boldsymbol{k})e^{i(E_i-\omega_{\boldsymbol{k}})t} + \{\omega_{\boldsymbol{k}} \leftrightarrow -\omega_{\boldsymbol{k}}\} \quad (43)$$

where the initial energy $E_i = E(\boldsymbol{p}_1) + E(\boldsymbol{q}_1)$.

- Similarly, for the V_{nf} element we get

$$V_{nf} = \bar{u}(p_1')\gamma^\mu v(q_1')\frac{\epsilon_\mu}{\sqrt{2\omega_{\boldsymbol{k}}}}\delta^{(3)}(\boldsymbol{p}_1'+\boldsymbol{q}_1'-\boldsymbol{k})e^{i(\omega_{\boldsymbol{k}}-E_f)t} \quad (44)$$

Computation of the red diagram

- As a result, we get:

$$M_{if} = \int dn \frac{V_{in}V_{nf}}{E_i - E_n}$$
$$= e^{i(E_i-E_f)t}\delta^{(3)}(\boldsymbol{p}_1' + \boldsymbol{q}_1' - \boldsymbol{p}_1 - \boldsymbol{q}_1)\frac{(\bar{v}(q_1)\gamma^\mu u(p_1))(\bar{u}(p_1')\gamma^\mu v(q_1'))}{E_i^2 - \omega_{\boldsymbol{k}}^2}\bigg|_{\boldsymbol{k}=\boldsymbol{p}_1+\boldsymbol{q}_1} \quad (45)$$

- …here $E_i = E(\boldsymbol{p}_1) + E(\boldsymbol{q}_1)$ – initial energy of electron+positron, $\boldsymbol{k} = \boldsymbol{p}_1 + \boldsymbol{q}_1$ – their total momentum of the pair

- …again one can show that $E_i^2 - \omega_{\boldsymbol{k}}^2 = k^2$ where k is a 4-momentum of the intermediate photon, $k = p_1 + q_1$

Total matrix element

- Amplitudes of two processes (blue and red on the Figure) should be **added together** before $|\dots|^2$ is taken. That is the probability of the process is proportional to $|\mathcal{M}_1+\mathcal{M}_2|^2$ rather than $|\mathcal{M}_1|^2+|\mathcal{M}_2|^2$ (interference terms are present)

$$|\mathcal{M}|^2 = (\dots)\left|\underbrace{i\frac{\left(\bar{u}(p_1')\gamma_\mu u(p_1)\right)\left(\bar{v}(q_1)\gamma^\mu v(q_1')\right)}{(p_1-p_1')^2}}_{\text{blue diagram}} - \underbrace{i\frac{\left(\bar{u}(p_1')\gamma_\mu v(q_1')\right)\left(\bar{v}(q_1)\gamma^\mu u(p_1)\right)}{(p_1+q_1)^2}}_{\text{red diagram}}\right|^2 \quad (46)$$

where $\dots$ is a prefactor, depending on energies/masses of particles.

- Perturbative series in $\hat{V}$

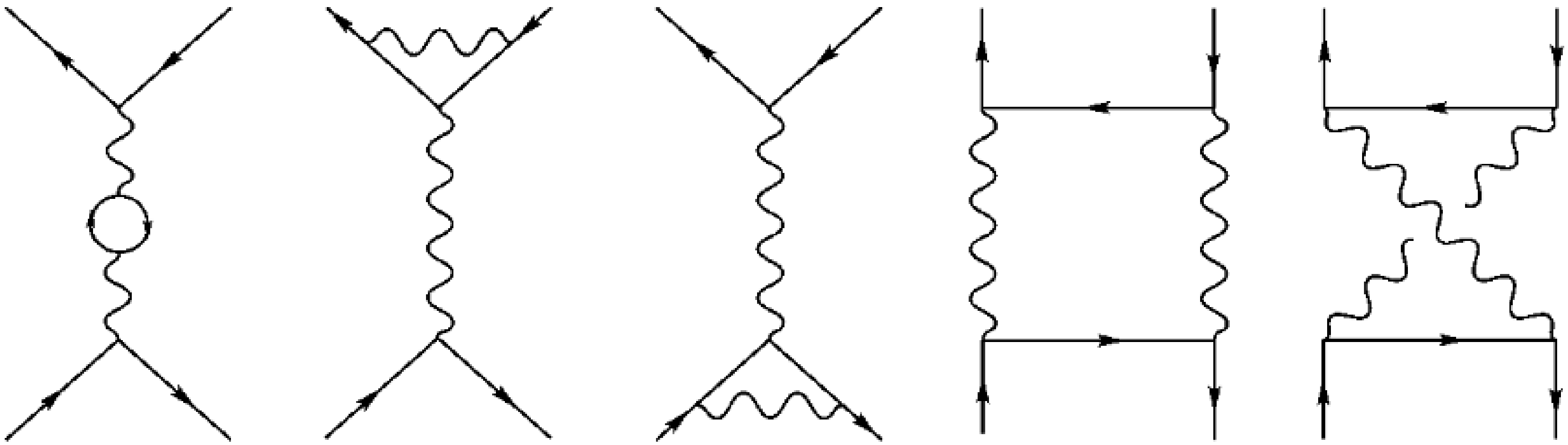
$$M_{if} = M_{if}^{(1)} + M_{if}^{(2)} + \dots$$

- We saw that $M_{if}^{(1)}$ is equal to zero (Eq. (1)). If non-zero, $M_{if}^{(1)} \propto e$ (charge in $\hat{V}$)

Total matrix element

- We found $M_{if}^{(2)}$ (2nd order perturbation theory in $\hat{V}$)
- This expression $M_{if}^{(2)}$ is proportional to e^2
- This parameter e is known experimentally to be small. The expansion parameter of electron-photon interaction is known as the **fine structure constant** $\alpha \equiv \frac{e^2}{\hbar c} \approx \frac{1}{137}$
- Indeed, consider next order in expansion. It includes more intermediate states – higher order in e

Next order



- There are e^+e^- scattering processes that go through **several** intermediate states:

$$M_{if}^{(3)} = \int dn_1 \int dn_2 \frac{V_{in_1} V_{n_1 n_2} V_{n_2 f}}{(E_i - E_{n_1})(E_{n_1} - E_{n_2})}$$

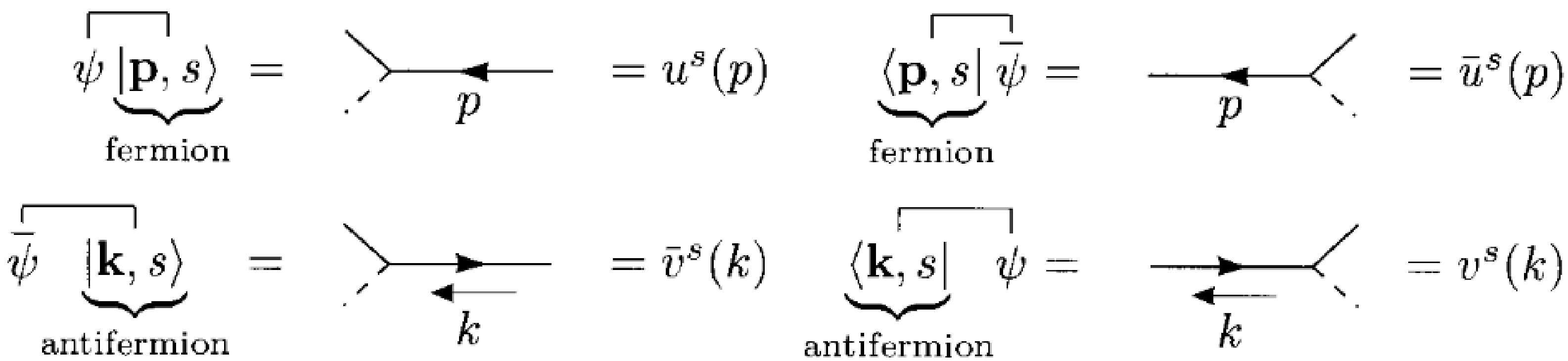
$$M_{if}^{(4)} = \int dn_1 \int dn_2 \int dn_3 \frac{V_{in_1} V_{n_1 n_2} V_{n_2 n_3} V_{n_3 f}}{(E_i - E_{n_1})(E_{n_1} - E_{n_2})(E_{n_2} - E_{n_3})}$$

Virtual particles

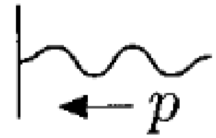
- In the above computations we have explicitly separate space and time.
- The intermediate states were "physical" (energy and momentum were related via $E^2 = \boldsymbol{p}^2 + m^2$), but only 3-momentum conservation was imposed in every vertex. The total (initial - final) energy was conserved, but for intermediate states it was not
- It is possible to construct explicitly Lorentz-invariant technique for computation of such matrix elements
- This is called **Feynman technique**. Its rules are presented in

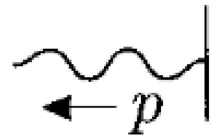
- There are three types of objects in constructing Feynman graphs:
  - external lines (real particles)
  - internal lines (virtual particles)
  - vertices (interaction points)

- To each **external fermion line** one associates a spinor u, v , etc. according to the following rule:



- To each **external photon line** one associates a polarization vector

External photon lines: $\overset{\sqcap}{A_\mu \, |\mathbf{p}\rangle} =$  $\mu = \epsilon_\mu(p)$

$\overset{\sqcap}{\langle \mathbf{p}| \, A_\mu} = \mu$  $= \epsilon^*_\mu(p)$

- Each virtual line adds a propagator:
  - Virtual fermion:

$$S_F(p) = \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$$

  - Virtual photon:

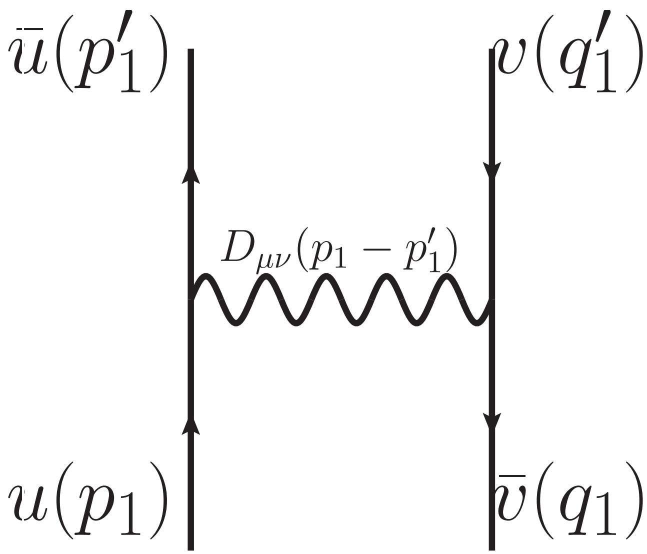
$$D_{\mu\nu}(p) = \frac{-i\eta_{\mu\nu}}{p^2 + i\epsilon}$$

- Each vertex (two fermion lines plus one photon line) receives a factor $-ie\gamma_\mu$

Feynman rules

- **Energy-momentum conservation** is imposed at every vertex

- Let us repeat the computation of electron-positron scattering

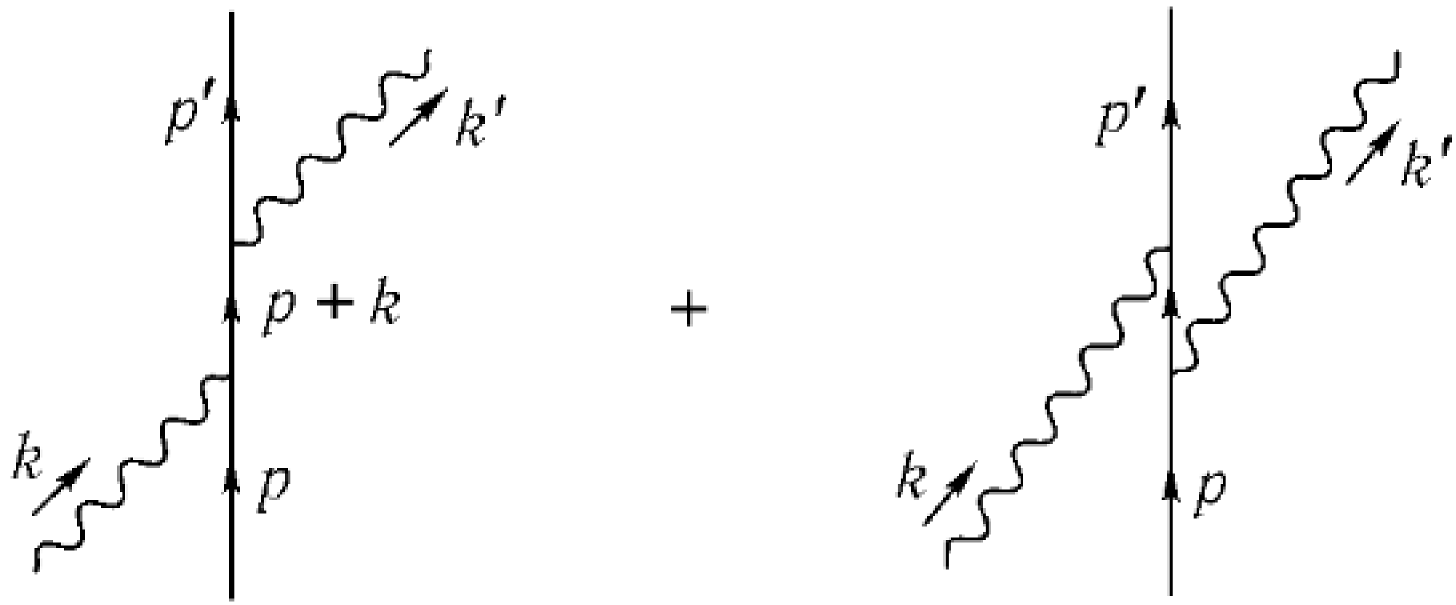


$$\mathcal{M} = \frac{\left(\bar{u}(p_1')(-ie\gamma_\mu)u(p_1)\right)\left(\bar{v}(q_1)(-ie\gamma^\mu)v(q_1')\right)}{(p_1 - p_1')^2}$$

- Similarly

$$\mathcal{M} = \frac{\left(\bar{u}(p_1')(-ie\gamma_\mu)v(q_1')\right)\left(\bar{v}(q_1)(-ie\gamma^\mu)u(p_1)\right)}{(p_1 + q_1)^2}$$

Compton scattering



- Write general form of matrix elements for Compton scattering
- Derive differential cross-section in non-relativistic case (photon energy $\omega \ll m_e$) and in ultra-relativistic case ($\omega \gg m_e$)

Peskin & Schroeder, Sec. 5.5

Pair creation

- Compute cross-section of $\gamma + \gamma \to e^+ + e^-$ (pair production)
- Derive differential cross-section in ultra-relativistic case ($\omega \gg m_e$)

Peskin & Schroeder, Sec. 5.5

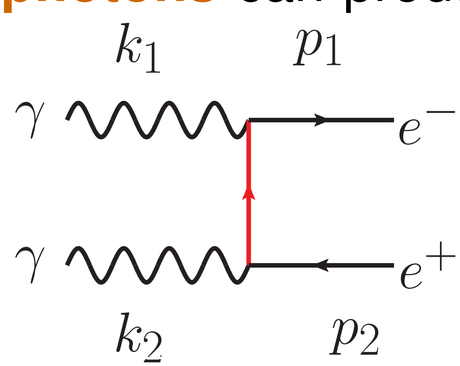
Onther consequences of Dirac theory of positrons

- Photons are **bosons** (particles of spin = 1). Electrons/positrons are **fermions** particles of spin = 1/2. Therefore, angular momentum conservation means that **photon couples to electron + positron**

- Photons could produce electron-positron pairs. **However**, the process $\boxed{\gamma \to e^+e^-}$ is not possible if all particle are "real" (i.e. photon obeys $E = cp$, electron/positron $E = \sqrt{p^2c^2 + m_e^2c^4}$ – "**on-shell conditions**")

- Instead, a **pair of photons** can produce electron-positron pair via

$\boxed{\gamma + \gamma \to e^+e^-}$:



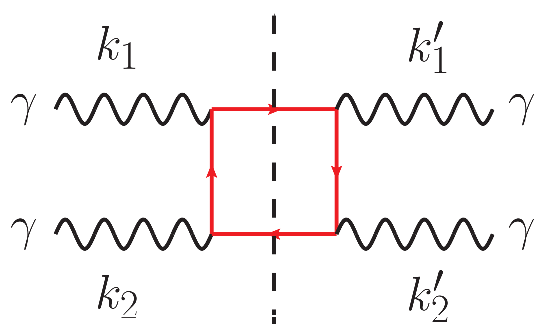
where $(k_1 + k_2)^2 \geq 4m_e^2$

- Similarly, electron-positron pair can **annihilate** into a pair of

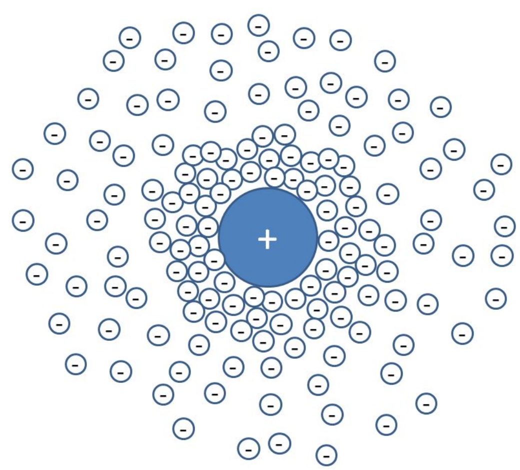
Onther consequences of Dirac theory of positrons

photons

- Kinematically, the red electron is **virtual** (i.e. for it $E \neq \sqrt{p^2c^2 + m^2c^4}$ – **check this**)

- 

  If energies of incoming photons are smaller than twice the electron mass (i.e. $(k_1 + k_2)^2 < 4m_e^2$) photons produce only virtual electron-positron pair which can then "annihilate" into another pair of photons – **light-on-light scattering**



- Charge screening:

Systems with many particles

- Presence of the negative-energy levels means that you can create particle-antiparticle pairs out of “nowhere”

- Particles in the pair can be real, but they can be also virtual (i.e. $E^2 - \boldsymbol{p}^2 \neq m^2$)

- According to the Heisenberg uncertainty relation $\Delta E\ \Delta t\ \gtrsim\ 1$, if one measures the state of system two times, separated by a short period $\Delta t \ll 1/m$, one will find a state with $1, 2, 3, ...$ additional pairs.

- It means that we no longer work with definite number of particles: number of particles may change! (Contrary to non-relativistic quantum mechanics)

- We need an approach that naturally takes into account states with different number of particles (we will return to this point in this Lecture)

Birth of quantum field theory

- Quantum electrodynamics (QED) – first quantum field theory. Free fields with interaction treated **perturbatively** in *fine-structure constant:* $\alpha = \frac{e^2}{\hbar c}$
- Divergencies? Many answers beyond tree-level (1st order) perturbation theory were infinite, because one had to sum up contributions from infinite number of **virtual particles** with growing energies $E_p = \pm\sqrt{m^2 + p^2}$