Lecture 4

Alexey Boyarsky

October 6, 2015

1 Conservation laws and symmetries

1.1 Ignorable Coordinates

During the motion of a mechanical system, the $2s$ quantities q_i and $\dot{q}_i$, $(i = 1, 2, \ldots, s)$ which specify the state of the system vary with time. There exist, however, functions of these quantities whose values remain constant during the motion, and depend only on the initial conditions. Such functions are called integrals of the motion.

Let us define the generalized momenta conjugate to the generalized coordinate q_k as

$$p_k = \frac{\partial L}{\partial q_k} \tag{1}$$

Lagrange's equations for a conservative system can then be written as:

$$\dot{p}_k = \frac{\partial L}{\partial q_k} \tag{2}$$

It is now readily apparent that if the Lagrangian does not explicitly contain the coordinate then

$$\dot{p}_k = 0 \quad \Rightarrow \quad p_k(t) = constant \tag{3}$$

The missing coordinate called ignorable, and its conjugate momentum is an integral of motion, so this can be a method of searching integrals of motion. But this method work only in limited number of situations.

The number of independent integrals of the motion for a closed mechanical system with s degrees of freedom is $2s - 1$ (see paragraph 6 from [3]). This is evident from the following simple arguments. The general solution of the equations of motion contains $2s$ arbitrary constants. Since the equations of motion for a closed system do not involve the time explicitly, the choice of the origin of time is entirely arbitrary, and one of the arbitrary constants in the solution of the equations can always be taken as an additive constant t_0 in the time. Eliminating $t+t_0$ from the $2s$ functions $q_i = q_i(t+t_0, C_1, C_2, \ldots, C_{2s-1})$, $\dot{q}_i = \dot{q}_i(t+t_0, C_1, C_2, \ldots, C_{2s-1})$, we can express the $2s-1$ arbitrary constants $C_1, C_2, \ldots, C_{2s-1}$ as functions of q and $\dot{q}$, and these functions will be integrals of the motion.

1.2 Energy

Not all integrals of the motion, however, are of equal importance in mechanics. There are some whose constancy is of profound significance, deriving from the fundamental homogeneity and isotropy of space and time. The quantities represented by such integrals of the motion are said to be conserved, and have an important common property of being additive: their values for a system composed of several parts whose interaction is negligible are equal to the sums of their values for the individual parts.

It is to this additivity that the quantities concerned owe their especial importance in mechanics. Let us suppose, for example, that two bodies interact during a certain interval of time. Since each of the additive integrals of the whole system is, both before and after the interaction, equal to the sum of its values for the two bodies separately, the conservation laws for these quantities immediately make possible various conclusions regarding the state of the bodies after the interaction, if their states before the interaction are known.

Let us consider first the conservation law resulting from the homogeneity of time. By virtue of this homogeneity, the Lagrangian of a closed system does not depend explicitly on time. The total time derivative of the Lagrangian can therefore be written

$$\frac{dL}{dt} = \sum_i \frac{\partial L}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i \tag{4}$$

If L depended explicitly on time, a term $\partial L/\partial t$ would have to be added on the right-hand side. Replacing $\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)$ in accordance with Lagrange's equations we obtain

$$\frac{dL}{dt} = \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i = \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) \tag{5}$$

or

$$\frac{d}{dt}\left(\sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L\right) = 0 \tag{6}$$

Hence we see that the quantity

$$\boxed{E = \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L} \tag{7}$$

remains constant during the motion of a closed system, i.e. it is an integral of the motion; it is called the energy of the system. The additivity of the energy follows immediately from that of the Lagrangian, since (7) shows that it is a linear function of the latter.

The Lagrangian of a closed system (or one in a constant field) is of the form $L = T(q, \dot{q}) - U(q)$, where T is a quadratic function of the velocities. Using Euler's theorem on homogeneous functions, we have

$$\sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i = \sum_i \frac{\partial T}{\partial \dot{q}_i}\dot{q}_i = 2T \tag{8}$$

Substituting this in (7) gives

$$E = T(q, \dot{q}) + U(q) \tag{9}$$

as it should be.

As an example, let us consider the Lagrangian, describing relativistic particle:

$$L\left[\frac{dx^\mu}{d\tau}\right] = -mc^2\sqrt{\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} \tag{10}$$

where $\eta_{\mu\nu}$ is the diagonal matrix $(1,-1,-1,-1)$ and $x^\mu(\tau)$ and $\dot{x}^\mu(\tau) \equiv \frac{dx^\mu}{d\tau}$ are used as generalized coordinates and velocities. We build energy (7) as

$$E = \sum_\mu \frac{\partial L}{\partial \dot{x}^\mu(\tau)}\dot{x}^\mu(\tau) - L \tag{11}$$

Notice, that the Lagrangian (10) is a *homogeneous function of degree one*, i.e.

$$L\Big[\lambda\dot{x}^\mu(\tau)\Big] = \lambda L\Big[\dot{x}^\mu(\tau)\Big] \tag{12}$$

This means that

$$\sum_\mu \frac{\partial L}{\partial \dot{x}^\mu(\tau)}\dot{x}^\mu(\tau) = L \tag{13}$$

and therefore the energy of the system, $E = 0$!

Now let us consider the situation when we have chosen $\vec{v} = d\vec{r}/dt$ as generalized velocities (and $\vec{r}$ as generalizde coordinates). In this case we can use the Lagrangian in the form:

$$L = -mc^2\sqrt{1-\frac{\vec{v}^2}{c^2}} \tag{14}$$

and we have

$$E = \frac{\partial L}{\partial \vec{v}}\vec{v} - L = -mc^2\left(\frac{1}{c^2}\frac{\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}\vec{v} - L\right) = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \tag{15}$$

— the total energy of the free particle of mass m.

1.3 Momentum

A second conservation law follows from the homogeneity of space (see paragraph 7 form [3]). By virtue of this homogeneity, the mechanical properties of a closed system are unchanged by any parallel displacement of the entire system in space. Let us therefore consider an infinitesimal displacement $\vec{\varepsilon}$, and obtain the condition for the Lagrangian to remain unchanged.

A parallel displacement is a transformation in which every particle in the system is moved by the same amount, the radius vector $vecr$ becoming $\vec{r}+\vec{\varepsilon}$. The change in L resulting from an infinitesimal change in the coordinates, the velocities of the particles remaining fixed, is

$$\delta L = \sum_k \frac{\partial L}{\partial \vec{r}_k}\delta\vec{r}_k = \vec{\varepsilon}\sum_k \frac{\partial L}{\partial \vec{r}_k} \tag{16}$$

where the summation is over the particles in the system. Since $\vec{\varepsilon}$ is arbitrary, the condition $\delta L = 0$ is equivalent to

$$\sum_k \frac{\partial L}{\partial \vec{r}_k} = 0 \tag{17}$$

From Lagrange's equations we therefore have

$$\sum_k \frac{d}{dt}\frac{\partial L}{\partial \vec{v}_k} = \frac{d}{dt}\sum_k \frac{\partial L}{\partial \vec{v}_k} = 0 \tag{18}$$

Thus we conclude that, in a closed mechanical system, the vector

$$\vec{P} = \sum_k \frac{\partial L}{\partial \vec{v}_k} \tag{19}$$

remains constant during the motion; it is called the momentum of the system. Differentiating the Lagrangian, we find that the momentum is given in terms of the velocities of the particles by

$$\vec{P} = \sum_k m_k \vec{v}_k \tag{20}$$

(this is actually true for any Lagrangian of the form $L = T(\vec{v}) - U(\vec{r})$).

For the **relativistic particle**, differentiating (14), i.e.

$$L = -mc^2\sqrt{1 - \frac{\vec{v}^2}{c^2}}$$

we find

$$\vec{P} = \frac{m\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{21}$$

It is also instructing to find a component of "momentum", for the generalized velocity $\dot{x}^0(\tau)$. Differentiating (10) with respect to $\dot{x}^0(\tau)$ we find

$$P_0 = \frac{\partial L}{\partial \dot{x}^0(\tau)} = \frac{mc^2}{\sqrt{1 - \left(\frac{\dot{\vec{x}}(\tau)}{\dot{x}^0(\tau)}\right)^2}} \tag{22}$$

The expression

$$\left(\frac{\dot{\vec{x}}(\tau)}{\dot{x}^0(\tau)}\right)^2 = \left(\frac{d\vec{x}(\tau)}{dx^0(\tau)}\right)^2 = \frac{\vec{v}^2}{c^2} \tag{23}$$

and we see that the momentum, corresponding to the generalized coordinate $x^0(\tau)$ is the energy of the particle.

The additivity of the momentum is evident. Moreover, unlike the energy, the momentum of the system is equal to the sum of its values $\vec{p}_a = m_a\vec{v}_a$ for the individual particles, whether or not the interaction between them can be neglected.

The three components of the momentum vector are all conserved only in the absence of an external field. The individual components may be conserved even in the presence of a field, however, if the potential energy in the field does not depend on all the Cartesian coordinates. The mechanical properties of the system are evidently unchanged by a displacement along the axis of a coordinate which does not appear in the potential energy, and so the corresponding component of the momentum is conserved. For example, in a uniform field in the z-direction, the x and y components of momentum are conserved.

1.4 Angular momentum

Let us now derive the conservation law which follows from the isotropy of space (see paragraph 9 from [3]). This isotropy means that the mechanical properties of a closed system do not vary when it is rotated as a whole in any manner in space. Let us therefore consider an infinitesimal rotation of the system, and obtain the condition for the Lagrangian to remain unchanged.

We shall use the vector $\vec{\varphi}$ of the infinitesimal rotation, whose magnitude is the angle of rotation φ, and whose direction is that of the axis of rotation (the direction of rotation being that of a right-handed screw driven along $\vec{\varphi}$).

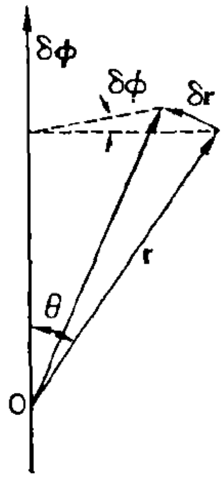

Figure 1: Vector rotation

Let us find, first of all, the resulting increment in the radius vector from an origin on the axis to any particle in the system undergoing rotation. The linear displacement of the end of the radius vector is related to the angle by $|\delta\vec{r}| = r \sin\theta\delta\varphi$ (fig. 1). The direction of $\delta\vec{r}$ is perpendicular to the plane of $\vec{r}$ and $\delta\vec{\varphi}$. Hence it is clear that

$$\delta\vec{r} = \delta\vec{\varphi} \times \vec{r} \tag{24}$$

When the system is rotated, not only the radius vectors but also the velocities of the particles change direction, and all vectors are transformed in the same manner. The velocity increment relative to a fixed system of coordinates is

$$\delta\vec{v} = \delta\vec{\varphi} \times \vec{v} \tag{25}$$

If these expressions are substituted in the condition that the Lagrangian is unchanged by the rotation:

$$\delta L = \sum_k \left(\frac{\partial L}{\partial \vec{r}_k} \delta \vec{r}_k + \frac{\partial L}{\partial \vec{v}_k} \delta \vec{v}_k \right) \tag{26}$$

and the derivative $\frac{\partial L}{\partial \vec{v}_k}$ replaced by $\vec{p}_k$, and $\frac{\partial L}{\partial \vec{r}_k}$ by $\dot{\vec{p}}_k$ the result is

$$\delta L = \sum_k \left(\dot{\vec{p}}_k \cdot \delta \vec{\varphi} \times \vec{r} + \vec{p}_k \delta \vec{\varphi} \times \vec{v} \right) = \delta \vec{\varphi} \sum_k \left(\vec{r} \times \dot{\vec{p}}_k + \vec{v} \times \vec{p}_k \right) = \delta \vec{\varphi} \frac{d}{dt} \sum_k \vec{r} \times \vec{p}_k \tag{27}$$

Since $\delta \vec{\varphi}$ is arbitrary, it follows that $\frac{d}{dt} \sum_k \vec{r} \times \vec{p}_k = 0$, and we conclude that the vector

$$\vec{M} = \sum_k \vec{r} \times \vec{p}_k \tag{28}$$

called the angular momentum or moment of momentum of the system, is conserved in the motion of a closed system. Like the linear momentum, it is additive, whether or not the particles in the system interact.

There are no other additive integrals of the motion. Thus every closed system has seven such integrals: energy, three components of momentum, and three components of angular momentum.

Although the law of conservation of all three components of angular momentum (relative to an arbitrary origin) is valid only for a closed system, the law of conservation may hold in a more restricted form even for a system in an external field. It is evident from the above derivation that the component of angular momentum along an axis about which the field is symmetrical is always conserved, for the mechanical properties of the system are unaltered by any rotation about that axis. Here the angular momentum must, of course, be defined relative to an origin lying on the axis.

The most important such case is that of a centrally symmetric field or central field, i.e. one in which the potential energy depends only on the distance from some particular point (the center). It is evident that the component of angular momentum along any axis passing through the center is conserved in motion in such a field. In other words, the angular momentum $\vec{M}$ is conserved provided that it is defined with respect to the center of the field.

Another example is that of a homogeneous field in the z-direction; in such a field, the component M_z of the angular momentum is conserved, whichever point is taken as the origin.

2 Motion with constraints

2.1 Solvable constraints

In general, if N particles are free to move in three-dimensional space but their $3N$ coordinates are connected by m conditions of constraint, then there exist $n = 3N - m$ independent generalized coordinates sufficient to describe uniquely the position of the N

particles and n independent degrees of freedom available for the motion, provided the constraints are of the type described in the preceding examples (see 10.2 of [1]). Such constraints are called *holonomic*. They must be expressible as equations of the form

$$f_i(\vec{x}_1, \ldots, \vec{x}_N, t) = 0 \qquad i = 1, \ldots, m \tag{29}$$

These equations are equalities, they are integrable in form, and they may or may not be explicitly time-dependent.[1]

Constraints that cannot be expressed as equations of equality or that are nonintegrable in form are called nonholonomic, and the equations representing such constraints cannot be used to eliminate from consideration any dependent coordinates describing the configuration of the system. As an example of such a constraint, consider a particle constrained to remain outside the surface of a sphere. (Humans on Earth capable of going to the moon but incapable of going more than a few miles underground represent a reasonable approximation to this situation.) This condition of constraint is given by the inequality

$$x^2 + y^2 + z^2 \geq R^2 \tag{30}$$

Clearly, this equation cannot be used to reduce below three the required number of independent coordinates of the particle when it lies outside the sphere. Because it is difficult to handle such situations using Lagrangian mechanics, we ignore them in this text.

When we have the system with holonomic constraints, the simplest way to solve it is to choose such generalized coordinates which will take into account all constraints and solves them automatically (for example θ and φ angles for particle on the sphere, or z and φ for a cylinder). See examples 10.6.1 and 10.6.2 in [1].

2.2 Forces of Constraint: Lagrange Multipliers

Even though the physical system has been subject to holonomic constraints, nowhere in our calculations have we had to consider specifically the forces that result from those constraints (see 10.7 of [1]). This is one of the great virtues of the Lagrangian method: direct inclusion of such forces of constraint in a solution for the motion of the constrained body is superfluous and, therefore, ignored. Suppose for some reason or another, however, we wish to know the values of those forces.

We can explicitly include the forces of constraint in the Lagrangian formulation if we so choose. In essence, this can be accomplished by not immediately invoking any equation(s) of constraint to reduce the number of degrees of freedom in a problem. Thus right from the outset we keep all the generalized coordinates in which the kinetic and potential energies of the system in question are expressed. This leads to more Lagrange equations for the problem, one for each additional generalized coordinate not eliminated by an equation of constraint. But because the coordinates are not independent, the resulting Lagrange equations as previously derived cannot be independent either. They can, however, be made independent through the technique of Lagrange multipliers (see 10.7 of [1]).

[1] The simplest example of non-holonomic constraints is $\dot{q} = \lambda = const$. In this case we know that $q = \lambda t + q_0$, but we cannot reduce the number of degrees of freedom, using this constraint.

For the sake of simplicity, we consider a system described by only two generalized coordinates q_1 and q_2 that are connected by a single equation of constraint

$$f(q_1, q_2, t) = 0 \tag{31}$$

We start with the Hamilton variational principle

$$\delta \int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2} \sum_i \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i dt \tag{32}$$

Only now, the δq_i are not independent; a variation in q_1 leads to a variation in q_2 consistent with the constraint given by equation (31), which, because it is fixed at any instant of time t, obeys the condition

$$\delta f = \frac{\partial f}{\partial q_1} \delta q_1 + \frac{\partial f}{\partial q_2} \delta q_2 = 0 \tag{33}$$

Solving for δq_2 in terms of δq_1

$$\delta q_2 = -\frac{\partial f / \partial q_1}{\partial f / \partial q_2} \delta q_1 \tag{34}$$

and on substituting into equation (32), we obtain

$$\int_{t_1}^{t_2} \left[\left(\frac{\partial L}{\partial q_1} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_1} \right) - \left(\frac{\partial L}{\partial q_2} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_2} \right) \frac{\partial f / \partial q_1}{\partial f / \partial q_2} \right] \delta q_1 \tag{35}$$

Only a single coordinate, q_1, is varied in this expression, and because it can be varied at will, the term in brackets must vanish. Thus,

$$\frac{\dfrac{\partial L}{\partial q_1} - \dfrac{d}{dt} \dfrac{\partial L}{\partial \dot{q}_1}}{\partial f / \partial q_1} = \frac{\dfrac{\partial L}{\partial q_2} - \dfrac{d}{dt} \dfrac{\partial L}{\partial \dot{q}_2}}{\partial f / \partial q_2} \equiv -\lambda(t) \tag{36}$$

The term on the left side is a function only of generalized coordinate q_1 and its derivative, while that on the right is a function only of q_2 and its derivative.[2] Furthermore, each term is implicitly time-dependent through these variables and possibly explicitly as well. The only way they can be equal at all times throughout the motion is if they are equal to a single function of time, which we call $-\lambda(t)$. Thus, we have

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} + \lambda(t) \frac{\partial f}{\partial q_i} = 0 \tag{37}$$

The same equations ((31) and (37)) can be obtained from Lagrangian

$$L' = L(q_i, \dot{q}_i, t) + \lambda(t) f(q_i) \tag{38}$$

as a Lagrange equations for coordinates q_i and λ. If there are more than one constraint, one simply adds additional terms, each with its own Lagrange multiplier, to the Lagrangian.

[2] This is obviously the case for the free system. For a general case see [2], §2.4.

We now have a system (37), described by three unknown functions of time: $q_1(t)$, $q_2(t)$, and $\lambda(t)$. We have three independent equations relating then: two Lagrangian equations of motion (37) and an equation of constraint (31). Additional terms in Eqs. (38) are *generalized forces of constraint*. We note that these generalized forces are not always forces but can also be torques when the corresponding generalized coordinates are angular.

$$Q_i = \lambda(t)\frac{\partial f}{\partial q_i} \tag{39}$$

that appear in the two Lagrangian equations of motion (37) are the forces of constraint that we desire. They appear in the problem only because we did not initially invoke the equation of constraint to reduce the degrees of freedom.

To understand the physical meaning of constraints, consider the following example, Eq. (37). Suppose we have a system of one particle in 3D in an external potential $V(\mathbf{x})$, then this can be rewritten as

$$\dot{\mathbf{p}} = -\nabla V(\mathbf{x}) + \lambda(t)\nabla g(\mathbf{x}).$$

On the lhs is the change in momentum, on the rhs are the forces, the first being the force from the external potential, the second the force of constraint that ensures that always $g(\mathbf{x}) = 0$. This force acts perpendicular to the surface on which the particle is allowed to move. For instance, for a pendulum one has $g(\mathbf{x}) = x^2 + y^2 + z^2 - l^2 = 0$ and $\nabla g(\mathbf{x})$ points indeed in the radial direction. The multiplier $\lambda(t)$ makes sure that the strength of the force, $|\lambda(t)\nabla g(\mathbf{x})|$, has always the right value to ensure the constraint.

3 Examples of systems with constraints

3.1 Tension of the rod of a simple pendulum

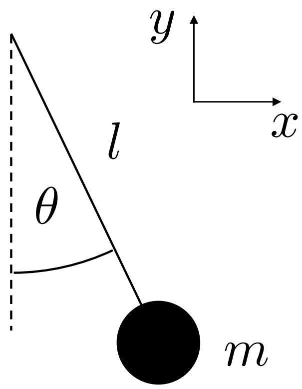

Figure 2: The pendulum.

A simple example for , the pendulum, see fig. 2. We choose polar coordinates so that the position of the mass is given by $(x(t), y(t)) = (r(t)\sin\theta(t), r(t)\cos\theta(t))$. The constraint is $g(r) = r - l = 0$. The kinetic energy is

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

and the potential energy is $V = -mgr\cos\theta$. The new Lagrangian, eq. 38, is now

$$L'\left(\theta, \dot{\theta}, r, \dot{r}, \lambda\right) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\cos\theta + \lambda\left(r - l\right).$$

We have now 3 Euler-Lagrange equations, one for θ (which turns out to be unimportant for our purpose), one for λ (which is just the constraint itself, see above) and one for r:

$$\frac{\partial L'}{\partial r} - \frac{d}{dt}\frac{\partial L'}{\partial \dot{r}} = mr\dot{\theta}^2 + mg\cos\theta + \lambda - m\ddot{r} = 0$$

from which follows (using $\ddot{r} = 0$):

$$\lambda = -mr\dot{\theta}^2 - mg\cos\theta.$$

The generalized force of constraint is

$$Q_r = \lambda\frac{\partial g}{\partial r} = \lambda = -mr\dot{\theta}^2 - mg\cos\theta.$$

This is the force that the massless rod has to sustain. Not surprisingly it is the sum of the centrifugal force and the radial component of the weight of the mass m.

3.2 A bead sliding on a uniformly rotating wire in a force-free space

The wire is straight, and is rotating uniformly around the z axis perpendicular to the wire. The transformation equations explicitly contain the time.

$$\begin{cases} x = r\cos\omega t \\ y = r\sin\omega t \end{cases} \tag{40}$$

Lagrangian of the system is equal to kinetic energy:

$$T = \frac{1}{2}m(\dot{r}^2 + \omega^2 r^2) \tag{41}$$

Note that T is not a homogeneous quadratic function of the generalized velocities,since there is now an additional term not involving r. The equation of motion is then

$$m\ddot{r} - mr\omega^2 = 0 \tag{42}$$

or

$$\ddot{r} = r\omega^2 \tag{43}$$

which is the familiar simple harmonic oscillator equation with a change of sign.The solution $r = C_1 e^{\omega t} + C_2 e^{-\omega t}$ shows that in general case the bead moves exponentially outward because of the centripetal acceleration. Let $C_1 = 1$, $C_2 = 0$. Angular momentum M is given by

$$M(t) = mr^2\omega^2 e^{\omega t} \tag{44}$$

The change of angular momentum is due to the *torque*

$$\frac{d\vec{M}}{dt} = \vec{Q} \times \vec{r} \tag{45}$$

(vector of angular momentum, $\vec{M}$ grows perpendicularly to the axis of rotation). This means that the *force of constraint*, Q, is given by

$$Q = mr\omega^3 e^{\omega t} \tag{46}$$

acting perpendicular to the wire and the axis of rotation.

4 Electromagnetic field

Consider Lagrangian of the electromagnetic field:

$$L = \frac{1}{2}(\vec{E}^2 - \vec{B}^2) \tag{47}$$

The Bianchi identities:

$$\nabla \vec{B} = 0 \tag{48}$$

and

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{49}$$

are solved identically if we express $\vec{E}$ and $\vec{B}$ as functions of A_0 and $\vec{A}$

$$\begin{cases} \vec{E} = -\vec{\nabla} A^0 - \dfrac{\partial \vec{A}}{\partial t} \\ \vec{B} = \vec{\nabla} \times \vec{A} \end{cases} \tag{50}$$

and the action

$$S = \int dt \int d^3\vec{x}\, L \tag{51}$$

Find the Lagrang equations varying the action (51) with respect to A_0 and $\vec{A}$.

Let us rewrite the Lagrangian (47) in terms of A_0 and $\vec{A}$:

$$L = \frac{1}{2}\dot{\vec{A}}^2 - \frac{1}{2}(\nabla \times \vec{A})^2 + \vec{\nabla} A_0 \cdot \left(\underbrace{\vec{\nabla} A^0 + \frac{\partial \vec{A}}{\partial t}}_{\equiv -\vec{E}} \right) \tag{52}$$

Putting this into the action (52) and integrating by parts, we find the following action:

$$S = \int dt \int d^3x \, \frac{1}{2} \left[\dot{\vec{A}}^2 - (\vec{\nabla} \times \vec{A})^2 \right] + A_0(\vec{\nabla} \cdot \vec{E}) \tag{53}$$

This equation has the form of the usual action for a variable $\vec{A}$ with Lagrangian $L = T - U$ where $T = \frac{1}{2}\dot{\vec{A}}^2$ and $U(\vec{A}) = (\vec{\nabla} \times \vec{A})^2$. The last term has the form of Lagrangian constraint with a Lagrange multiplier A_0. Varying with respect to A_0 we get the Gauss law:

$$\vec{\nabla}\cdot\vec{E} = 0 \tag{54}$$

Let us build the energy of the electromagnetic field. As L does not depend on $\dot{A}_0$, we have

$$\text{Energy} = \int d^3x \left[\frac{\delta L}{\delta\dot{\vec{A}}(x)}\dot{\vec{A}}(x)\right] - L \tag{55}$$

the variation over $\dot{\vec{A}}$ is given by

$$\delta_{\dot{\vec{A}}}\int d^3x\,\vec{E}^2 = \frac{2}{c}\int d^3x\,\vec{E}\cdot\delta\dot{\vec{A}} \tag{56}$$

Therefore we find that

$$\text{Energy} = \int d^3x\left[-\frac{1}{c}\vec{E}\cdot\dot{\vec{A}} - \frac{1}{2}\vec{E}^2 + \frac{1}{2}\vec{B}^2\right] \tag{57}$$

and finally we should notice that

$$\int d^3x\frac{1}{c}\vec{E}\cdot\dot{\vec{A}} = -\int d^3x\vec{E}^2 \tag{58}$$

as the term $\int d^3x\vec{E}\cdot\nabla A_0 = 0$ due to $\nabla\cdot\vec{E} = 0$.

References

[1] Analytical Mechanics, G.R. Fowles and G.L. Cassiday 6th or 7th edition (Thomson Learning, inc., 1999), ISBN 9780534408138.

[2] Classical Mechanics (3rd Edition) by H. Goldstein, C.P. Poole Jr., J.L. Safko, ISBN 978-0201657029

[3] Course of theoretical physics, v.1 Mechanics, 3rd edition, by L.D. Landau and E.M. Livshitz ISBN 0 7506 2896 0