Lecture 4

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1 Examples of Lagrangians

Until now we were defining Lagrangians as $L = T - U$. Let us now take an opposite approach. To describe a physical system, we will *start* with a Lagrangian (using considerations of symmetries and other physical properties of the system in hand) and *define* a theoretical model of this system via Lagrangian. Given this Lagrangian we can then derive equations of motion, energy of the system, etc. and then check that our model describes the system correctly. As Lagrangian is only one function (depending probably on some unknown parameters), this approach may be easier than to deduce equations of motion directly from the experiment.

1.1 Reminder: non-relativistic particle

Describing physical systems, we have a freedom to choose a coordinate system and reference frame (see §2 from [3]). If we were to choose an arbitrary frame of reference, space could be inhomogeneous and anisotropic for us. This means that, even if a body interacted with no other bodies, its various positions in space and its different orientations would not be mechanically equivalent. For example, imagine that you are sitting in the rotating frame. Then the laws of motion of a free body depends on how close you are to the axis of rotation (e.g. centrifugal force depends on the distance). This means that the space is not homogeneous for you!.

The same would in general be true of time, which would likewise be inhomogeneous; that is, different instants would not be equivalent – if, for example, the frame that you are using changes its velocity at some moment t_0, a free body that was in rest before t_0 will begin to move in some direction at the next instant, even if it is not subject to any external action.

Such properties of space and time would evidently complicate the description of mechanical phenomena.

However, our experience, summarised by the first Newton's law of mechanics, tells us that we can always choose a frame of reference such that space is homogeneous and isotropic and time is homogeneous. Such a frame is called an *inertial frame*. In particular, in such a frame a free body which is at rest at some instant remains always at rest.

We can now draw from this consideration some immediate inferences concerning the form of the Lagrangian of a particle, moving freely, in an inertial frame of reference. The homogeneity of space and time implies that the Lagrangian cannot contain explicitly either the radius vector $\vec{r}$ of the particle or the time t, i.e. L must be a function of the velocity $\vec{v}$ only. Since space is isotropic, the Lagrangian must also be independent of the direction of $\vec{v}$, and is therefore a function only of its magnitude, i.e. of $\vec{v}^2 = v^2$:

$$L = L(v^2) \tag{1}$$

Since the Lagrangian is independent of $\vec{r}$, we have $\partial L/\partial\vec{r} = 0$, and so Lagrange's equation is

$$\frac{d}{dt}\frac{\partial L}{\partial \vec{v}} = 0$$

whence $\partial L/\partial\vec{v} =$ constant. Since $\partial L/\partial\vec{v}$ is a function of the velocity only, it follows that

$$\vec{v} = \text{constant} \tag{2}$$

Thus we conclude that, in an inertial frame, any free motion takes place with a velocity which is constant in both magnitude and direction. This is the law of inertia.

If we consider, besides the inertial frame, another frame moving uniformly in a straight line relative to the inertial frame, then the laws of free motion in the other frame will be the same as in the original frame: free motion takes place with a constant velocity.

Experiment shows that not only are the laws of free motion the same in the two frames, but the frames are entirely equivalent in all mechanical respects. Thus there is not one but an infinity of inertial frames moving, relative to one another, uniformly in a straight line. In all these frames the properties of space and time are the same, and the laws of mechanics are the same. This constitutes Galileo's relativity principle, one of the most important principles of mechanics.

Let us now go on to determine the form of the Lagrangian, and consider first of all the simplest case, that of the free motion of a particle relative to an inertial frame of reference (see §4 from [3]). As we have already seen, the Lagrangian in this case can depend only on the square of the velocity. To discover the form of this dependence, we make use of Galileo's relativity principle. If an inertial frame K is moving with an infinitesimal velocity $\vec{\varepsilon}$ relative to another inertial frame K', then

$$\vec{v}' = \vec{v} + \vec{\varepsilon}. \tag{3}$$

Since the equations of motion must have the same form in every frame, the Lagrangian $L(v^2)$ must be converted by this transformation into a function L' which differs from $L(v^2)$, if at all, only by the total time derivative of a function of coordinates and time (see the end of the previous lecture or paragraph 2 from [3]).

We have $L' = L(v'^2) = L(v^2 + 2\vec{v}\cdot\vec{\varepsilon} + \vec{\varepsilon}^2)$. Expanding this expression in powers of $\vec{\varepsilon}$ and neglecting terms above the first order, we obtain

$$L'(v'^2) = L(v^2) + \frac{\partial L}{\partial v^2} 2\vec{v}\cdot\vec{\varepsilon} \tag{4}$$

We demand that the action does not change under the change of the references frame (3). Therefore the Lagrangian $L'(v'^2)$ can differ from the Lagrangian $L(v^2)$ by *total time derivative only*. The second term on the right of this equation is a total time derivative only if it is a linear function of the velocity $\vec{v}$. Indeed, let us try to find a function $F(\vec{r}, t)$ such that

$$\frac{dF(\vec{r}, t)}{dt} = \frac{\partial L}{\partial v^2} 2\vec{v} \tag{5}$$

Explicitly differentiating $F(\vec{v}, \vec{r})$ we find

$$\frac{dF(\vec{r},t)}{dt} = \frac{\partial F}{\partial \vec{r}}\dot{\vec{r}} + \frac{\partial F}{\partial t} \tag{6}$$

Comparing (6) with the right hand side of Eq. (5) we see that $\frac{\partial F}{\partial t} = 0$ (because it cannot contain speed) and

$$F(\vec{r}) = 2\frac{\partial L}{\partial v^2}\vec{r} \tag{7}$$

where $\partial L/\partial v^2$ is independent of the velocity, i.e. the Lagrangian is in this case proportional to the square of the velocity, and we write it as

$$L = \frac{1}{2}mv^2 \tag{8}$$

1.2 Free relativistic particle[1]

Let us try to construct a Lagrangian (and an action) for a free relativistic particle. The action should be Lorentz-invariant, ie. a scalar. As in the case of non-relativistic particle, homogeneity and isotropy of space (and time in this case) means that the action cannot depend on $x^\mu = (t, \vec{r})$. It should depend on the derivatives $dx^\mu/d\tau$ where τ is any parametrization of the world-line of a particle. The only such invariant, which is first order in derivatives is the *interval* ds:

$$ds = \sqrt{\big(dx^0(\tau)\big)^2 - \big(d\vec{r}(\tau)\big)^2} \tag{9}$$

and the action

$$S = -\alpha \int ds \tag{10}$$

(where α is some constant to be determined).

Let us choose time t as a parameter along the world line (i.e. $x^0 = ct$). Then the interval (9) becomes

$$ds = \sqrt{c^2 - \left(\frac{d\vec{r}}{dt}\right)^2}\,dt \tag{11}$$

where $\frac{d\vec{r}}{dt} = \vec{v}$ – 3-dimensional velocity. Substituting expression (11) into the action (10), one gets

$$S = -\alpha \int \sqrt{c^2 - \vec{v}^2}dt \tag{12}$$

Comparing Eq. (12) with $S = \int L dt$ we find that the Lagrangian of the relativistic particle has the form

$$L = -\alpha c\sqrt{1 - \frac{\vec{v}^2}{c^2}} \tag{13}$$

To find the value of α, we can take the non-relativistic limit $v \ll c$ and find that the Lagrangian becomes

$$L \approx -\alpha c + \frac{\alpha v^2}{2c}, \qquad v \ll c \tag{14}$$

[1]This is an additional section, not included in the homework or exam

The constant term does not change the Lagrangian equations, therefore we see that α should be identified with the mass $\alpha = mc$ and finally we obtain

$$L = -mc^2\sqrt{1 - \frac{\vec{v}^2}{c^2}} \tag{15}$$

As we will see below this is an example of a Lagrangian that does not have the form $L = T - U$

1.3 Lagrangian of the electromagnetic field[2]

Let us try to build the Lagrangian of electromagnetic field. The Lagrangian should be gauge invariant therefore it should be built of vectors $\vec{E}$ and $\vec{B}$. The Lagrangian should be a scalar, and it should be at most 2nd order in electric/magnetic field (otherwise one would not get the linear Maxwell's equations). There are three scalars that can be built out of $\vec{E}$ and $\vec{B}$:

$$\text{scalars} \quad : \quad \vec{E}\cdot\vec{B}, \quad \vec{E}^2, \quad \vec{B}^2 \tag{16}$$

It is important to remind here that in fact $\vec{E}$ and $\vec{B}$ are not independent!

Let us start with $\vec{E}\cdot\vec{B}$. The expression

$$\int dt \int d^3x\, \vec{E}\cdot\vec{B} = \int dt \frac{\partial}{\partial t}\left(\int d^3x\, \vec{A}\cdot\vec{B}\right) \tag{17}$$

i.e. the Lagrangian $\int d^3x\, \vec{A}\cdot\vec{B}$ is the total time derivative and does not change the equations of motion! We conclude therefore that the Lagrangian of the electromagnetic field will have a general form

$$L_{EM} = \int d^3x \left[\alpha\vec{E}^2 + \beta\vec{B}^2\right] \tag{18}$$

There are several ways to fix the constants α, β

1. In Section 2.2 we will build *energy* of the system, starting from the Lagrangian. We will see if (and only of) $\alpha = -\beta$ the energy is *positivly definite* and coincides with the known form of the electromagnetic energy:

$$E_{EM} = \frac{1}{2}\int d^3x \left[\vec{E}^2 + \vec{B}^2\right] \tag{19}$$

2. For the same choice, $\alpha = -\beta$, the Lagrangian (18) acquires additional symmetry – it becomes invariant under Lorentz transformations. Indeed, in the case it can be re-written in terms of $F_{\mu\nu}$ (previous Homework!) and becomes explicitly Lorentz invariant (a 3+1 dimensional rather then 3 dimensional scalar).

3. For the same choice of $\alpha = -\beta$ the equations of motions for this Lagrangian coincide with the correct Maxwell's equations.

[2]Additional material, will not be asked at exam

1.4 Example: a non-relativistic particle in the electromagnetic field.

A single particle of mass m and charge q moving at a velocity, $\vec{v} = \dot{\vec{r}}$ in external electric $\vec{E}(\vec{r}, t)$ and magnetic $\vec{B}(\vec{r}, t)$ fields, which may depend upon time and position. The charge experiences a force, called the **Lorentz force**, given by (see 1.5 of [2])

$$m\,\ddot{\vec{r}} = q\left(\vec{E} + [\dot{\vec{r}} \times \vec{B}]\right) \tag{20}$$

To see that Eq. (20) is a set of Lagrange equations is a non-trivial task. We will demonstrate how the Lagrangian approach helps here.

Lagrangian of a particle in the electromagnetic field. Let us start consider the action of a system of a particle and of the electromagnetic fields.

Our idea will be the following: we will consider the action for

$$S\Big[\vec{r}(t), \vec{v}(t)\,|\phi(t,\vec{x}), \vec{A}(t,\vec{x})\Big] = \int dt \Big[L_{\text{particle}}[\vec{r},\vec{v}] + L_{\text{EM field}}[\phi,\vec{A}] + L_{\text{interaction}}l[\vec{r}(t), \vec{v}(t)\,|\phi(t,\vec{x}), \vec{A}(t,\vec{x})\Big]\Big] \tag{21}$$

The Lagrangian of a free particle is just

$$L_{\text{particle}} = \frac{m}{2}\dot{\vec{r}}^2 \tag{22}$$

The variation of the action (21) with respect to ϕ and $\vec{A}$ will give the Maxwell's equations. The interaction term $L_{\text{interaction}}$ will give the right hand side of the Maxwell's equations:

$$\frac{\partial F_{\mu\nu}}{\partial x^{\nu}} = \frac{\delta L_{\text{interaction}}}{\delta A_{\mu}} \tag{23}$$

Recall that when studying the Lagrangian of the electromagnetic field we learned that the correct *generalized coordinates* for the electromagnetic field are the components of the vector potential $A_\mu = (\phi, \vec{A})$ rather than $\vec{E}$ and $\vec{B}$ themselves.

$$\vec{E} = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t} \tag{24}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{25}$$

In more details: the Lagrangian of electro-magnetic field is

$$L_{\text{EM field}} = \frac{1}{2}\int d^3\vec{r}\Big[\vec{E}^2 - \vec{B}^2\Big] \tag{26}$$

The interaction part can be determined in the following form. We know that the Maxwell's equation with the *sources* have the form

$$\begin{aligned} &\text{Lagrange eqn. for } \phi \quad : \quad \nabla\cdot\vec{E} = \rho \qquad\qquad \equiv -\frac{\delta L_{\text{interaction}}}{\delta\phi} \\ &\text{Lagrange eqn. for } \vec{A} \quad : \quad \nabla\times\vec{B} - \frac{\partial\vec{E}}{\partial t} = \vec{\jmath} \ \equiv -\frac{\delta L_{\text{interaction}}}{\delta\vec{A}} \end{aligned} \tag{27}$$

Looking at Eqs. (27) we see that we should find a term $L_{\text{interaction}}$ such that

$$\frac{\delta L_{\text{interaction}}}{\delta \vec{A}} = \vec{\jmath} \quad \text{and} \quad \frac{\delta L_{\text{interaction}}}{\delta \phi} = \rho \tag{28}$$

In case of a point particle with the charge q is located at the point in space $\vec{r}(t)$ at the time t[3]

$$\vec{\jmath}(\vec{r},t) = \rho(\vec{r},t)\vec{v} \tag{29}$$

and

$$L_{\text{interaction}} = \int d^3\vec{r}\left[-\rho\phi + \vec{A}\cdot\vec{\jmath}\right] \tag{30}$$

Plugging (29) into (30) we conclude that for the particle in the *external* electromagnetic field we have the Lagrangian

$$L = \frac{mv^2}{2} - q\varphi + q\vec{A}\cdot\vec{v} \tag{31}$$

Let's check, that using Lagrangian (31) we have right equations of motion:

$$\frac{\partial L}{\partial \vec{r}} = -q\vec{\nabla}\varphi + q\vec{\nabla}(\vec{A}\cdot\vec{v}) \tag{32}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \vec{v}} = m\ddot{r} + q\frac{\partial \vec{A}}{\partial t} + q\sum_i \frac{\partial \vec{A}}{\partial x_i}\frac{\partial x_i}{\partial t} = m\ddot{r} + q\frac{\partial \vec{A}}{\partial t} + q(\vec{v}\cdot\vec{\nabla})\vec{A} \tag{33}$$

so we have (using (24)):

$$m\ddot{r} = -q\vec{\nabla}\varphi + q\vec{\nabla}(\vec{A}\cdot\vec{v}) - q\frac{\partial \vec{A}}{\partial t} - q(\vec{v}\cdot\vec{\nabla})\vec{A} = q\vec{E} + q\left[\vec{\nabla}(\vec{A}\cdot\vec{v}) - (\vec{v}\cdot\vec{\nabla})\vec{A}\right] \tag{34}$$

Now let's use vector identity

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B}) \tag{35}$$

and rewrite the second term in (20):

$$q\vec{v}\times\vec{B} = q\vec{v}\times(\vec{\nabla}\times\vec{A}) = q\vec{\nabla}(\vec{v}\cdot\vec{A}) - q(\vec{v}\cdot\vec{\nabla})\vec{A} \tag{36}$$

Comparing (34) and (36) we see, that we obtain eq. (20), so our Lagrangian was constructed correctly.

2 Conservation laws and symmetries

2.1 Ignorable Coordinates

During the motion of a mechanical system, the $2s$ quantities q_i and $\dot{q}_i$, $(i = 1, 2, \ldots, s)$ which specify the state of the system vary with time. There exist, however, functions of

[3] The correct mathematical expression $\rho(\vec{r},t) = q\delta^{(3)}(\vec{r} - \vec{r}(t))$

these quantities whose values remain constant during the motion, and depend only on the initial conditions. Such functions are called integrals of the motion.

Let us define the generalized momenta conjugate to the generalized coordinate q_k as

$$p_k = \frac{\partial L}{\partial q_k} \tag{37}$$

Lagrange's equations for a conservative system can then be written as:

$$\dot{p}_k = \frac{\partial L}{\partial q_k} \tag{38}$$

It is now readily apparent that if the Lagrangian does not explicitly contain the coordinate then

$$\dot{p}_k = 0 \quad \Rightarrow \quad p_k(t) = constant \tag{39}$$

The missing coordinate called ignorable, and its conjugate momentum is an integral of motion, so this can be a method of searching integrals of motion. But this method work only in limited number of situations.

The number of independent integrals of the motion for a closed mechanical system with s degrees of freedom is $2s - 1$ (see paragraph 6 from [3]). This is evident from the following simple arguments. The general solution of the equations of motion contains $2s$ arbitrary constants. Since the equations of motion for a closed system do not involve the time explicitly, the choice of the origin of time is entirely arbitrary, and one of the arbitrary constants in the solution of the equations can always be taken as an additive constant t_0 in the time. Eliminating $t+t_0$ from the $2s$ functions $q_i = q_i(t+t_0, C_1, C_2, \ldots, C_{2s-1})$, $\dot{q}_i = \dot{q}_i(t+t_0, C_1, C_2, \ldots, C_{2s-1})$, we can express the $2s-1$ arbitrary constants $C_1, C_2, \ldots, C_{2s-1}$ as functions of q and $\dot{q}$, and these functions will be integrals of the motion.

2.2 Energy

Not all integrals of the motion, however, are of equal importance in mechanics. There are some whose constancy is of profound significance, deriving from the fundamental homogeneity and isotropy of space and time. The quantities represented by such integrals of the motion are said to be conserved, and have an important common property of being additive: their values for a system composed of several parts whose interaction is negligible are equal to the sums of their values for the individual parts.

It is to this additivity that the quantities concerned owe their especial importance in mechanics. Let us suppose, for example, that two bodies interact during a certain interval of time. Since each of the additive integrals of the whole system is, both before and after the interaction, equal to the sum of its values for the two bodies separately, the conservation laws for these quantities immediately make possible various conclusions regarding the state of the bodies after the interaction, if their states before the interaction are known.

Let us consider first the conservation law resulting from the homogeneity of time. By virtue of this homogeneity, the Lagrangian of a closed system does not depend explicitly on time. The total time derivative of the Lagrangian can therefore be written

$$\frac{dL}{dt} = \sum_i \frac{\partial L}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i \tag{40}$$

If L depended explicitly on time, a term $\partial L/\partial t$ would have to be added on the right-hand side. Replacing $\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)$ in accordance with Lagrange's equations we obtain

$$\frac{dL}{dt} = \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i = \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) \tag{41}$$

or

$$\frac{d}{dt}\left(\sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L\right) = 0 \tag{42}$$

Hence we see that the quantity

$$\boxed{E = \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L} \tag{43}$$

remains constant during the motion of a closed system, i.e. it is an integral of the motion; it is called the energy of the system. The additivity of the energy follows immediately from that of the Lagrangian, since (43) shows that it is a linear function of the latter.

The Lagrangian of a closed system (or one in a constant field) is of the form $L = T(q,\dot{q}) - U(q)$, where T is a quadratic function of the velocities. Using Euler's theorem on homogeneous functions, we have

$$\sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i = \sum_i \frac{\partial T}{\partial \dot{q}_i}\dot{q}_i = 2T \tag{44}$$

Substituting this in (43) gives

$$E = T(q,\dot{q}) + U(q) \tag{45}$$

as it should be.

As an example, let us consider the Lagrangian (10) describing relativistic particle:

$$L\left[\frac{dx^\mu}{d\tau}\right] = -mc^2\sqrt{\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} \tag{46}$$

where $\eta_{\mu\nu}$ is the diagonal matrix $(1,-1,-1,-1)$ and $x^\mu(\tau)$ and $\dot{x}^\mu(\tau) \equiv \frac{dx^\mu}{d\tau}$ are used as generalized coordinates and velocities. We build energy (43) as

$$E = \sum_\mu \frac{\partial L}{\partial \dot{x}^\mu(\tau)}\dot{x}^\mu(\tau) - L \tag{47}$$

Notice, that the Lagrangian (46) is a *homogeneous function of degree one*, i.e.

$$L\Big[\lambda \dot{x}^\mu(\tau)\Big] = \lambda L\Big[\dot{x}^\mu(\tau)\Big] \tag{48}$$

This means that

$$\sum_\mu \frac{\partial L}{\partial \dot{x}^\mu(\tau)}\dot{x}^\mu(\tau) = L \tag{49}$$

and therefore the energy of the system, $E = 0$!

Now let us consider the situation when we have chosen $\vec{v} = d\vec{r}/dt$ as generalized velocities (and $\vec{r}$ as generalizde coordinates). In this case we can use the Lagrangian in the form (15)

$$L = -mc^2\sqrt{1 - \frac{\vec{v}^2}{c^2}} \tag{50}$$

and we have

$$E = \frac{\partial L}{\partial \vec{v}}\vec{v} - L = -mc^2\left(\frac{1}{c^2}\frac{\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}\vec{v} - L\right) = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \tag{51}$$

— the total energy of the free particle of mass m.

2.3 Momentum

A second conservation law follows from the homogeneity of space (see paragraph 7 form [3]). By virtue of this homogeneity, the mechanical properties of a closed system are unchanged by any parallel displacement of the entire system in space. Let us therefore consider an infinitesimal displacement $\vec{\varepsilon}$, and obtain the condition for the Lagrangian to remain unchanged.

A parallel displacement is a transformation in which every particle in the system is moved by the same amount, the radius vector $vecr$ becoming $\vec{r} + \vec{\varepsilon}$. The change in L resulting from an infinitesimal change in the coordinates, the velocities of the particles remaining fixed, is

$$\delta L = \sum_k \frac{\partial L}{\partial \vec{r}_k}\delta\vec{r}_k = \vec{\varepsilon}\sum_k \frac{\partial L}{\partial \vec{r}_k} \tag{52}$$

where the summation is over the particles in the system. Since $\vec{\varepsilon}$ is arbitrary, the condition $\delta L = 0$ is equivalent to

$$\sum_k \frac{\partial L}{\partial \vec{r}_k} = 0 \tag{53}$$

From Lagrange's equations we therefore have

$$\sum_k \frac{d}{dt}\frac{\partial L}{\partial \vec{v}_k} = \frac{d}{dt}\sum_k \frac{\partial L}{\partial \vec{v}_k} = 0 \tag{54}$$

Thus we conclude that, in a closed mechanical system, the vector

$$\vec{P} = \sum_k \frac{\partial L}{\partial \vec{v}_k} \tag{55}$$

remains constant during the motion; it is called the momentum of the system. Differentiating the Lagrangian, we find that the momentum is given in terms of the velocities of the particles by

$$\vec{P} = \sum_k m_k\vec{v}_k \tag{56}$$

For the **relativistic particle** we find, differentiating (50)

$$\vec{P} = \frac{m\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{57}$$

It is also instructing to find a component of "momentum", for the generalized velocity $\dot{x}^0(\tau)$. Differentiating (46) with respect to $\dot{x}^0(\tau)$ we find

$$P_0 = \frac{\partial L}{\partial \dot{x}^0(\tau)} = \frac{mc^2}{\sqrt{1 - \left(\frac{\dot{\vec{x}}(\tau)}{\dot{x}^0(\tau)}\right)^2}} \tag{58}$$

The expression

$$\left(\frac{\dot{\vec{x}}(\tau)}{\dot{x}^0(\tau)}\right)^2 = \left(\frac{d\vec{x}(\tau)}{dx^0(\tau)}\right)^2 = \frac{\vec{v}^2}{c^2} \tag{59}$$

and we see that the momentum, corresponding to the generalized coordinate $x^0(\tau)$ is the energy of the particle.

The additivity of the momentum is evident. Moreover, unlike the energy, the momentum of the system is equal to the sum of its values $\vec{p}_a = m_a\vec{v}_a$ for the individual particles, whether or not the interaction between them can be neglected.

The three components of the momentum vector are all conserved only in the absence of an external field. The individual components may be conserved even in the presence of a field, however, if the potential energy in the field does not depend on all the Cartesian coordinates. The mechanical properties of the system are evidently unchanged by a displacement along the axis of a coordinate which does not appear in the potential energy, and so the corresponding component of the momentum is conserved. For example, in a uniform field in the z-direction, the x and y components of momentum are conserved.

2.4 Angular momentum

Let us now derive the conservation law which follows from the isotropy of space (see paragraph 9 from [3]). This isotropy means that the mechanical properties of a closed system do not vary when it is rotated as a whole in any manner in space. Let us therefore consider an infinitesimal rotation of the system, and obtain the condition for the Lagrangian to remain unchanged.

We shall use the vector $\vec{\varphi}$ of the infinitesimal rotation, whose magnitude is the angle of rotation φ, and whose direction is that of the axis of rotation (the direction of rotation being that of a right-handed screw driven along $\vec{\varphi}$).

Let us find, first of all, the resulting increment in the radius vector from an origin on the axis to any particle in the system undergoing rotation. The linear displacement of the end of the radius vector is related to the angle by $|\delta\vec{r}| = r\sin\theta\delta\varphi$ (fig. 1). The direction of $\delta\vec{r}$ is perpendicular to the plane of $\vec{r}$ and $\delta\vec{\varphi}$. Hence it is clear that

$$\delta\vec{r} = \delta\vec{\varphi} \times \vec{r} \tag{60}$$

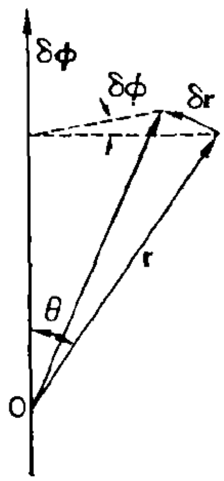

Figure 1: Vector rotation

When the system is rotated, not only the radius vectors but also the velocities of the particles change direction, and all vectors are transformed in the same manner. The velocity increment relative to a fixed system of coordinates is

$$\delta\vec{v} = \delta\vec{\varphi} \times \vec{v} \tag{61}$$

If these expressions are substituted in the condition that the Lagrangian is unchanged by the rotation:

$$\delta L = \sum_k \left(\frac{\partial L}{\partial \vec{r}_k} \delta\vec{r}_k + \frac{\partial L}{\partial \vec{v}_k} \delta\vec{v}_k \right) \tag{62}$$

and the derivative $\dfrac{\partial L}{\partial \vec{v}_k}$ replaced by $\vec{p}_k$, and $\dfrac{\partial L}{\partial \vec{r}_k}$ by $\dot{\vec{p}}_k$ the result is

$$\delta L = \sum_k \left(\dot{\vec{p}}_k \cdot \delta\vec{\varphi} \times \vec{r} + \vec{p}_k \delta\vec{\varphi} \times \vec{v} \right) = \delta\vec{\varphi} \sum_k \left(\vec{r} \times \dot{\vec{p}}_k + \vec{v} \times \vec{p}_k \right) = \delta\vec{\varphi} \frac{d}{dt} \sum_k \vec{r} \times \vec{p}_k \tag{63}$$

Since $\delta\vec{\varphi}$ is arbitrary, it follows that $\dfrac{d}{dt} \sum_k \vec{r} \times \vec{p}_k = 0$, and we conclude that the vector

$$\vec{M} = \sum_k \vec{r} \times \vec{p}_k \tag{64}$$

called the angular momentum or moment of momentum of the system, is conserved in the motion of a closed system. Like the linear momentum, it is additive, whether or not the particles in the system interact.

There are no other additive integrals of the motion. Thus every closed system has seven such integrals: energy, three components of momentum, and three components of angular momentum.

Although the law of conservation of all three components of angular momentum (relative to an arbitrary origin) is valid only for a closed system, the law of conservation may hold in a more restricted form even for a system in an external field. It is evident from the above derivation that the component of angular momentum along an axis about which the field is symmetrical is always conserved, for the mechanical properties of the system are unaltered by any rotation about that axis. Here the angular momentum must, of course, be defined relative to an origin lying on the axis.

The most important such case is that of a centrally symmetric field or central field, i.e. one in which the potential energy depends only on the distance from some particular point (the center). It is evident that the component of angular momentum along any axis passing through the center is conserved in motion in such a field. In other words, the angular momentum $\vec{M}$ is conserved provided that it is defined with respect to the center of the field.

Another example is that of a homogeneous field in the z-direction; in such a field, the component M_z of the angular momentum is conserved, whichever point is taken as the origin.

3 Symmetric form of the Lagrange equations of motion

In the Lagrangian formulation, a system with n degrees of freedom possesses n equations of motion of the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \tag{65}$$

As the equations are of second order, the motion of the system for all moments of time is determined when $2n$ initial values are specified: the n q_i's and n $\dot{q}_i$'s at a particular time t_1. Equations of motions in the Lagrangian form are written however as equations for n independent functions $q_i(t)$: we represent the state of the system by a point in an n-dimensional configuration space and follow how this point changes with time.

To make the evolution of the initial state $q; \dot{q}$ more explicit in the equations of motion and to simplify the solution of these equations, it seems useful to re-write the equations of motion as a system of $(2n)$ first order differential equation on the variables $q(t); \dot{q}(t)$.

Let us consider generalised velocities $\dot{q}_i$ as independent variables $y_i = \dot{q}_i$ and accelerations $\ddot{q}_i = \dot{y}_i$. Lagrangian system of equations then becomes

$$\begin{cases} \dot{q}_i = y_i \\ \dfrac{d}{dt}\dfrac{\partial L(q,y)}{\partial y_i} = \dfrac{\partial L}{\partial q_i}(q,y) \end{cases} \tag{66}$$

The second equation equation implicitly contains $\dot{y}_i$, but it is not resolved explicitly for it. To make it more explicit on can introduce the generalised momenta

$$p_i(t) = \frac{\partial L(q,\dot{q})}{\partial \dot{q}_i}(t) \tag{67}$$

such that the Lagrange equations simplify to become

$$\dot{p}_i = \frac{\partial L(q,\dot{q})}{\partial q_i} \tag{68}$$

This equation contain, however, there variables: q, p and $\dot{q}$. If we resolve Eqn (67) and express generalised velocities $\dot{q}_i$ as functions of q_i and p_i

$$\dot{q}_i = \dot{Q}_i(q,p), \qquad p_i = \left.\frac{\partial L(q,\dot{q},t)}{\partial \dot{q}_i}\right|_{\dot{q}_i=\dot{Q}_i(q,p)} \tag{69}$$

we will obtain a system of $2n$ first order equations for the independent variables q_i and p_i

$$\begin{cases} \dot{q}_i = \dot{Q}_i(q,p) \\ \dot{p}_i = P_i(q,p) \equiv \dfrac{\partial L(q,\dot{Q}(q,p),t)}{\partial q_i} \end{cases} \tag{70}$$

3.1 Example : particles in a potential field

$$L = \sum_i \frac{m\dot{x}_i^2}{2} - U(x) \qquad\qquad p_i = m\dot{x}_i \tag{71}$$

$$\begin{cases} \dot{x}_i = \dfrac{p_i}{m} \\ \dot{p}_i = -\dfrac{\partial U(x)}{\partial x_i} \end{cases} \tag{72}$$

We see that in this case p_i is just the particle's momentum ($p_i = m\dot{x}_i$) and the equations of motion are just the Newtons law in case of the potential force $F_i = -\dfrac{\partial U(x)}{\partial x_i}$:

$$\frac{dp_i}{dt} = F_i \tag{73}$$

Generalized momenta. Using Cartesian position coordinates as generalized coordinates, the Lagrangian can also be written as

$$L = \sum_i \frac{m\dot{x}_i{}^2}{2} - q\varphi + \sum_i qA_i\dot{x}_i \tag{74}$$

Using $p_i = \frac{\partial L}{\partial \dot{x}_i}$ we have equations:

$$p_i = m\dot{x}_i + qA_i(x) \tag{75}$$

$$\dot{p}_i = -q\frac{\partial \varphi}{\partial x_i} + q\sum_k \frac{\partial A_k}{\partial x_i}\dot{x}_k \tag{76}$$

Notice (Eq. (75)) that in this example, generalized momentum p_i is *not equal* kinetic momentum $m\dot{x}_i$!

We can express $\dot{x}_i$ via p_i and x_i:

$$\dot{x}_i = \frac{p_i - qA_i(x)}{m} \tag{77}$$

$$\dot{p}_i = -q\frac{\partial \varphi}{\partial x_i} + \frac{q}{m}\sum_k \frac{\partial A_k}{\partial x_i}(p_k - qA_k) \tag{78}$$

The system of Eqs. (77)–(78) is equivalent to Eq. (20).

References

[1] Analytical Mechanics, G.R. Fowles and G.L. Cassiday 6th or 7th edition (Thomson Learning, inc., 1999), ISBN 9780534408138.

[2] Classical Mechanics (3rd Edition) by H. Goldstein, C.P. Poole Jr., J.L. Safko, ISBN 978-0201657029

[3] Course of theoretical physics, v.1 Mechanics, 3rd edition, by L.D. Landau and E.M. Livshitz ISBN 0 7506 2896 0