Lecture 2

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1 Lagrange mechanics

In all the examples we considered above the we have seen that the state of the system was defined as a minimal set of data required to determine the subsequent evolution of the system. For many systems this set consists of a configuration (the analog of the position of a particle) and its time derivative (the analog of velocity), however there are also deviations from this picture defined by the dynamics of the system (its equations of motion). The nature of configuration may be different: this can be a discret set of variables or a number of continuous functions (e.g. vector field). More over, the same system can be rewritten in different coordinate systems (different parametrisations of configurations), depending for example on the external forces acting on this system.

In this Section we will try to find common feature of all the above examples and find a description of the evolution of a dynamical system that will have the same form for all these systems. As the particles' dynamics is the basic example for mechanics, we will start with rewriting the dynamics of such a system of particles in an arbitrary system of coordinates and will then generalize the resulting description for a wider class of systems.

Let q_i, $i = 1 \dots n$, are new co-ordinates and f is some function of coordinates and time $f = f(q_1, \dots, q_n, t)$.

We consider a system of particles with Cartesian co-ordinates $\vec{r}_1, \dots, \vec{r}_N$ and we want to change them to independent generalized co-ordinates $q_1, \dots, q_n$, $n \leq 3N$. The relations between old and new co-ordinates are:

$$\vec{r}_I = \vec{r}_I(q_1, \dots, q_n, t) \tag{1}$$

(index I numbers particles and runs from $1, \dots, N$). For each particle we have:

$$m_I \frac{d\vec{v}_I}{dt} = \vec{F}_I \tag{2}$$

1.1 Transformation of 2nd Newton's law under the change of variables (1)

Let us take a particle in the 3-dimensional oscillator potential. The equations of motion are:

$$m\ddot{\vec{r}} = -k\vec{r} \tag{3}$$

(here force $\vec{F} = -k\vec{r}$ — Hooke's law).

Let us make a general co-ordinate change from $x_i(t)$ to $q_j(t)$:

$$x_i = f_i\Big(q_1(t), q_2(t), q_3(t), t\Big) \tag{4}$$

$$\dot{x}_i = \sum_j \frac{\partial f_i}{\partial q_j}\dot{q}_j + \frac{\partial f_i}{\partial t} \tag{5}$$

$$\ddot{x}_i = \sum_{j,k} \frac{\partial^2 f_i}{\partial q_j \partial q_k}\dot{q}_j\dot{q}_k + \sum_j \frac{\partial f_i}{\partial q_j}\ddot{q}_j + \sum_j \frac{\partial^2 f_i}{\partial t \partial q_j}\dot{q}_j + \frac{\partial^2 f_i}{\partial t^2} \tag{6}$$

and equation of motion will be in form:

$$m\left(\frac{\partial f_i}{\partial q_1}\ddot{q}_1 + \frac{\partial f_i}{\partial q_2}\ddot{q}_2 + \frac{\partial f_i}{\partial q_3}\ddot{q}_3\right) = - kf_i - m\left(\sum_{j,k} \frac{\partial^2 f_i}{\partial q_j \partial q_k}\dot{q}_j\dot{q}_k + \sum_j \frac{\partial^2 f_i}{\partial t \partial q_j}\dot{q}_j + \frac{\partial^2 f_i}{\partial t^2}\right) \tag{7}$$

For example, for Galilei's transformation $x_i = q_i + v_i t$ equations don't become too difficult:

$$m\ddot{q}_i = -kq_i - kv_i t \tag{8}$$

But for rotating co-ordinate system with frequency ω (see 5.2 of [1]):

$$\begin{cases} x = x'\cos\omega t + y'\sin\omega t \\ y = - x'\sin\omega t + y'\cos\omega t \\ z = z' \end{cases} \tag{9}$$

we get:

$$\begin{cases} m\left(\ddot{x}'\cos\omega t + \ddot{y}'\sin\omega t\right) = (m\omega^2 - k)(x'\cos\omega t + y'\sin\omega t) - m\omega(-\dot{x}'\sin\omega t + \dot{y}'\cos\omega t) \\ m\left(-\ddot{x}'\sin\omega t + \ddot{y}'\cos\omega t\right) = (m\omega^2 - k)(-x'\sin\omega t + y'\cos\omega t) + m\omega(\dot{x}'\cos\omega t + \dot{y}'\sin\omega t) \\ m\ddot{z}' = - kz' \end{cases} \tag{10}$$

1.2 Covariant form

We saw in the previous Section that under a general coordinate transformation (1) the force $\vec{F}(\vec{r})$ transforms in a non-covariant way (Eq. (7)).

What object would remain invariant in all coordinate systems? Consider *work performed by forces*, $\vec{F}$:

$$\delta A \equiv \sum_I \vec{F}_I \delta\vec{r}_I \tag{11}$$

Under a general coordinate transformation (1) this expression can be rewritten as

$$\delta A = \sum_I \vec{F}_I\delta\vec{r}_I = \sum_j \left(\sum_I \vec{F}_I \frac{\partial \vec{r}_I}{\partial q_j}\right)\delta q_j = \sum_j Q_j \delta q_j \tag{12}$$

where we have introduced notation Q_j (called *generalized forces*):

$$Q_j \equiv \sum_{I=1}^{N} \vec{F}_I \cdot \frac{\partial \vec{r}_I}{\partial q_j} \tag{13}$$

The work A is *invariant* under the change of variables. To see what equation it satisfies, let us multiply equation (2) by $\partial \vec{r}_I / \partial q_j$ and add results:

$$\sum_{I=1}^{N} m_I \frac{d\vec{v}_I}{dt} \cdot \frac{\partial \vec{r}_I}{\partial q_j} = \sum_{I=1}^{N} \vec{F}_I \cdot \frac{\partial \vec{r}_I}{\partial q_j} \tag{14}$$

We see that the generalised force Q_j appears in the r.h.s. of the equation (14). As j changes from 1 to n we have n such equations.

$$\sum_{I=1}^{N} m_I \frac{d\vec{v}_I}{dt} \cdot \frac{\partial \vec{r}_I}{\partial q_j} = \frac{d}{dt} \sum_{I=1}^{N} m_I \vec{v}_I \cdot \frac{\partial \vec{r}_I}{\partial q_j} - \sum_{I=1}^{N} m_I \vec{v}_I \cdot \frac{d}{dt} \frac{\partial \vec{r}_I}{\partial q_j} \tag{15}$$

We can further rewrite this expression using two identities

$$\frac{\partial \vec{v}_I}{\partial \dot{q}_j} = \frac{\partial \vec{r}_I}{\partial q_j} \tag{16}$$

and

$$\frac{\partial \vec{v}_I}{\partial q_k} = \frac{d}{dt} \frac{\partial \vec{r}_I}{\partial q_k} \tag{17}$$

we have

$$\frac{d}{dt} \sum_{I=1}^{N} m_I \vec{v}_I \cdot \frac{\partial \vec{v}_I}{\partial \dot{q}_j} - \sum_{I=1}^{N} m_I \vec{v}_I \cdot \frac{\partial \vec{v}_I}{\partial q_j} = Q_j \tag{18}$$

Let's rewrite left part of (18) as

$$\frac{d}{dt} \frac{\partial}{\partial \dot{q}_j} \underbrace{\sum_{I=1}^{N} \frac{m_I v_I^2}{2}}_{\equiv T} - \frac{\partial}{\partial q_j} \sum_{I=1}^{N} \frac{m_I v_I^2}{2} = Q_j \tag{19}$$

By definition, the sum in left hand side is a kinetic energy of the system T:

$$T \equiv \sum_{I=1}^{N} \frac{m_I v_I^2}{2} \tag{20}$$

So, finally:

$$\boxed{\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j} \tag{21}$$

These n equations are called Lagrange equations. Q_j are called generalized forces. The main profit of this approach is that Lagrange equations are covariant — they doesn't change form after change of co-ordinate system.

Lagrange equations contains $n+1$ functions — n generalized forces Q_i and kinetic energy T. But they can be simplified for the case of potential forces $\vec{F}_I$. The potentiality of force means, that there exists scalar function of co-ordinates and time $V_I(\vec{r},t)$, that $\vec{F}_I = -\vec{\nabla} V_I$. So, if all forces are potential ones:

$$\begin{aligned} Q_j = \sum_{I=1}^{N} \vec{F}_I \frac{\partial \vec{r}_I}{\partial q_j} &= -\sum_{I=1}^{N} \left(\frac{\partial V_I}{\partial x_I}\frac{\partial x_I}{\partial q_j} + \frac{\partial V_I}{\partial y_I}\frac{\partial y_I}{\partial q_j} + \frac{\partial V_I}{\partial z_I}\frac{\partial z_I}{\partial q_j} \right) \\ &= -\sum_{I=1}^{N} \frac{\partial V_I}{\partial q_j} = -\frac{\partial U}{\partial q_j} \end{aligned} \tag{22}$$

where $U = \sum_I V_I$ — potential energy of the system. Therefore:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = -\frac{\partial U}{\partial q_j} \tag{23}$$

and, using fact that $\partial U/\partial \dot{q}_j = 0$, we finally have:

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0} \tag{24}$$

where

$$\boxed{L = T - U} \tag{25}$$

Function $L(q,\dot{q},t)$ is called *Lagrange function*, or just *Lagrangian* of the system (see 10.4 of [1]).

2 The principle of least action

What is the origin of the Lagrange equations of motion (24)? It turns out they can be obtained from conception called the *principle of least action* or Hamilton's principle (see 10.1 of [1]).

Namely, we consider a system whose state is characterized by a set of variables q_j. Its dynamics is determined via the Lagrangian function $L(q_1, q_2, \ldots, q_s, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, t)$, or briefly $L(q,\dot{q},t)$.

Consider a "trajectory" that starts at moment of time t_1 in the position $q^{(1)} = q(t_1)$ and ends at time t_2 in the position $q^{(2)} = q(t_2)$. We want to determine the minimum of the functional

$$S = \int\limits_{t_1}^{t_2} L(q,\dot{q},t)dt \tag{26}$$

The integral (26) is called *the action.*

Let us now derive the differential equations which solve the problem of minimizing the integral (26). For simplicity, we shall at first assume that the system has only one degree of freedom, so that only one function $q(t)$ has to be determined.

Let $q_{min} = q_{min}(t)$ be the function for which S is a minimum. This means that S is increased when $q(t)$ is replaced by any function of the form

$$q_{min}(t) + \delta q(t) \tag{27}$$

where $\delta q(t)$ is a function which is small everywhere in the interval of time from t_1 to t_2; (it has no connection to virtual displacement from previous section) and is called *a variation of the function* $q(t)$. Since, for $t = t_1$ and for $t = t_2$, all the functions (27) must take the values $q^{(1)}$ and $q^{(2)}$ respectively, it follows that

$$\delta q(t_1) = \delta q(t_2) = 0 \tag{28}$$

The change in S when q is replaced by $q + \delta q$ is

$$\int_{t_1}^{t_2} L(q + \delta q, \dot{q} + \delta \dot{q}, t)dt - \int_{t_1}^{t_2} L(q, \dot{q}, t)dt$$

When this difference is expanded in powers of δq and $\delta \dot{q}$ in the integrand, the leading terms are of the first order. The necessary condition for S to have a minimum is that these terms (called *the first variation*, or simply the *variation*, of the integral) should be zero. Thus the principle of least action may be written in the form

$$\delta S = \delta \int_{t_1}^{t_2} L(q, \dot{q}, t)dt = 0 \tag{29}$$

or, effecting the variation,

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) dt = 0$$

Since $\delta q = d\delta q/dt$, we obtain, on integrating the second term by parts,

$$\delta S = \frac{\partial L}{\partial \dot{q}} \delta q \bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \delta q dt \tag{30}$$

The conditions (28) show that the first term in (30) is zero. There remains an integral which must *vanish for all values of* $\delta q(t)$. This can be so only if the integrand is zero identically. Thus we have

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

When the system has more than one degree of freedom, the n different functions $q_i(t)$ must be varied independently in the principle of least action. We then evidently obtains equations of the form

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \tag{31}$$

We see, that principle of the least action gives the right equations of motion for the system of particles. In fact, the scope of it's application is much more wider and lies from elementary particle physics up to gravity and cosmology. So we will postulate it as a fundamental principle of mechanics. Writing Lagrangian is equivalent to determining all properties of the physical system.

One further general remark should be made. Let us consider two functions $L'(q, \dot{q}, t)$ and $L(q, \dot{q}, t)$, differing by the total derivative with respect to time of some function $f(q, t)$ of co-ordinates and time:

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{df(q, t)}{dt} \tag{32}$$

The integrals (26) calculated from these two functions are such that

$$S' = \int_{t_1}^{t_2} L'(q, \dot{q}, t)dt = \int_{t_1}^{t_2} L(q, \dot{q}, t)dt + \int_{t_1}^{t_2} \frac{df(q, t)}{dt}dt = S + f(q^{(2)}, t_2) - f(q^{(1)}, t_1)$$

i.e. they differ by a quantity which gives zero on variation, so that the conditions $\delta S' = 0$ and $\delta S = 0$ are equivalent, and the form of the equations of motion is unchanged. Thus the Lagrangian is defined only up to a total time derivative of any function of co-ordinates and time.

Note that we have derived the Lagrangian equations for arbitrary generalized coordinates q_i and therefore their form will remain *the same* under arbitrary change of coordinates.

From now on we will describe the dynamical system via its *Lagrangian*, introducing generalized coordinates, as dictated by symmetries of the problem.

3 Examples of Lagrangians

So far the Hamilton's principle looks as a very complicated way of obtaining the equation of motion. We will see below, that for more complicated systems, especially those that contain constraints, such a framework is extremely useful. Consider examples below

3.1 Pendulum

Consider a pendulum, a mass m attached to a massless rod of length l that is suspended from a pivot at position $(x, y) = (0, 0)$ around which it can swing freely. The potential of the mass in the gravitational field is given by mgy. The Lagrange function of the pendulum is thus given by

$$L(x, y, \dot{x}, \dot{y}) = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) - mgy. \tag{33}$$

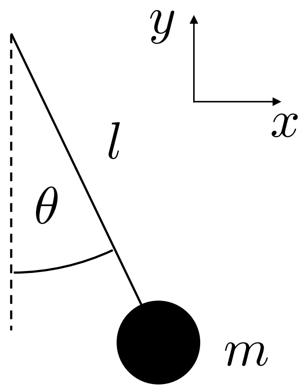

Figure 1: The pendulum.

The Euler-Lagrange equations for the x- and y-coordinates lead to two equations of motion, $\ddot{x} = 0$ and $\ddot{y} = -g$.

Unfortunately these equations are completely wrong. What we found are the equations of motion of a free particle in 2 dimensions in a gravitational field. Solutions are e.g. trajectories of rain drops or of cannon balls but certainly not the motion of a pendulum. What went wrong? We forgot to take into account the presence of the rod that imposes a constraint, namely that $x^2 + y^2 = l^2$. A better approach would be to use a coordinate system that accounts automatically for this constraint, namely to describe the state of the pendulum by the angle $\theta(t)$ between the pendulum and the y-direction, see Fig. 1. But how does the equation of motion look in terms of this angle?

Here comes into play a great advantage of Hamilton's principle: it is independent of the coordinate system that one chooses. Suppose one goes from one coordinate system x_1, x_2,..., x_N to another coordinate system q_1, q_2,..., q_f via the transformations $\mathbf{q} = \mathbf{q}(\mathbf{x})$ and $\mathbf{x} = \mathbf{x}(\mathbf{q})$.

The trajectory $\mathbf{x}(t)$ becomes then $\mathbf{q}(\mathbf{x}(t))$. The action functional can then be rewritten as

$$S[\mathbf{x}] = \int_{t_1}^{t_2} L(\mathbf{x}(t), \dot{\mathbf{x}}(t))\, dt = \int_{t_1}^{t_2} L\left(\mathbf{x}(\mathbf{q}(t)), \sum_{i=1}^{f} \frac{\partial \mathbf{x}(\mathbf{q}(t))}{\partial q_i} \dot{q}_i\right) dt. \tag{34}$$

Notice that the form (34) is possible only if $\mathbf{q} = \mathbf{q}(\mathbf{x})$ and does not depend explicitly on time

The rhs of Eq. 34 is again of the form

$$S[\mathbf{q}] = \int_{t_1}^{t_2} \tilde{L}(\mathbf{q}(t), \dot{\mathbf{q}}(t))\, dt \tag{35}$$

with a new Lagrangian $\tilde{L}$. Also here Hamilton's principle must hold, i.e., the dynamic evolution of the system follows from the Euler-Lagrange equations

$$\frac{\partial \tilde{L}}{\partial q_i} - \frac{d}{dt} \frac{\partial \tilde{L}}{\partial \dot{q}_i} = 0 \tag{36}$$

for $i = 1, ..., f$.

If we have a system with constraints we can sometimes introduce coordinates that automatically fulfill those constraints. The equations of motion are then simply given by the Euler-Lagrange equations in these coordinates. Let us go back to the pendulum. We describe now the configuration of the pendulum by the angle $\theta(t)$, see Fig. 1. In terms of this angle the kinetic energy of the pendulum is given by $ml^2\dot{\theta}^2/2$ and the potential energy by $-mlg\cos\theta$. This leads to the following Lagrange function:

$$L\left(\theta,\dot{\theta}\right)=\frac{ml^2}{2}\dot{\theta}^2+mgl\cos\theta. \tag{37}$$

The corresponding Euler-Lagrange equation is given by

$$\ddot{\theta}(t)=-\frac{g}{l}\sin\theta(t), \tag{38}$$

which is indeed the equation of motion of the pendulum.

3.2 Pendulum on a movable support

Consider a mass M that can move freely along a horizonal line without friction. Attached to the mass M is a pendulum of mass m via a massless connection of length l (Fig. 2). We calculate now the Lagrange equations for this system.

We first need to find a suitable coordinate system. The system has 2 degrees of freedom (you can find this number by subtracting the two constraints from the 4 degrees of freedom of the unconstrained masses). Practical coordinates are the position X of the mass M along the line and the angle θ between the pendulum and the direction of gravity. The position of the pendulum body is then given by

$$x=X+l\sin\theta \quad \text{and} \quad z=-l\cos\theta.$$

The kinetic energy is then given by:

$$T=\frac{1}{2}M\dot{X}^2+\frac{1}{2}m\left(\dot{x}^2+\dot{z}^2\right)=\frac{1}{2}M\dot{X}^2+\frac{1}{2}m\left[\left(\dot{X}+l\dot{\theta}\cos\theta\right)^2+\left(l\dot{\theta}\sin\theta\right)^2\right].$$

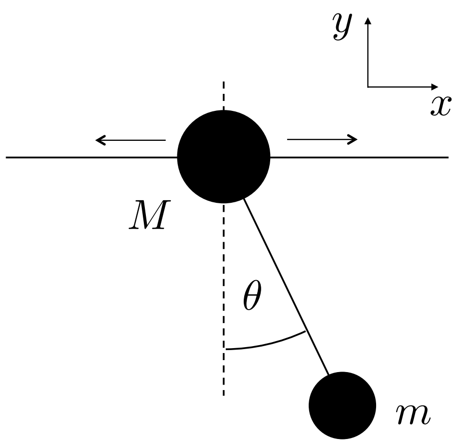

Figure 2: Example: a pendulum on a movable support.

This simplifies to

$$T = \frac{1}{2}(m+M)\dot{X}^2 + \frac{1}{2}m\left[2l\dot{X}\dot{\theta}\cos\theta + \left(l\dot{\theta}\right)^2\right].$$

The potential energy is given by

$$V = -mgl\cos\theta.$$

We can now obtain the equations of motions by taking derivatives of the Lagrangian with respect to the coordinates and to their time derivatives. This is done separately for the two coordinates. The Lagrange equation 36 for the coordinate X is given by:

$$\frac{d}{dt}\left[\frac{\partial(T-V)}{\partial\dot{X}}\right] - \frac{\partial(T-V)}{\partial X} = 0,$$

leading to

$$(m+M)\ddot{X} + ml\frac{d}{dt}\left(\dot{\theta}\cos\theta\right) = 0$$

or

$$(m+M)\ddot{X} = ml\left(\dot{\theta}^2\sin\theta - \ddot{\theta}\cos\theta\right).$$

Note that the partial derivative with respect to $\dot{X}$ is only taken on those places where this variable occurs but that the derivative with respect to the time t acts on all variables including θ en $\dot{\theta}$. Another point to note here is that quantity $\partial(T-V)/\partial\dot{X}$ is conserved (i.e. does not change with time). This follows always immediately if the Lagrangian does not depend on one of the coordinates (here X). You can check easily that this quantity is here the total momentum in the X-direction.

For the other coordinate, θ, we obtain:

$$\frac{d}{dt}\left[\frac{\partial(T-V)}{\partial\dot{\theta}}\right] - \frac{\partial(T-V)}{\partial\theta} = 0,$$

leading to

$$ml\left(l\ddot{\theta} + \ddot{X}\cos\theta - \dot{X}\dot{\theta}\sin\theta\right) + ml\dot{X}\dot{\theta}\sin\theta + mgl\sin\theta = 0$$

or

$$\ddot{\theta} + \frac{\ddot{X}}{l}\cos\theta + \frac{g}{l}\sin\theta = 0.$$

This example shows how straightforward the equations of motion can be derived with the Lagrange formalism as compared to deriving them from Newton's formalism which involves force vectors.

3.3 Particle in a central force field

For a particle in a central field the motion takes place in a plane (see 6.1 of [1]). We choose polar coordinates. The velocity has a radial and a tangential component. The kinetic and the potential energies are given by

$$T = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \quad \text{and} \quad V = V(r).$$

The Lagrange equation for r is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r} = \frac{\partial L}{\partial r} = mr\dot{\theta}^2 - \frac{dV}{dr}$$

and for θ:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m\frac{d}{dt}\left(r^2\dot{\theta}\right) = \frac{\partial L}{\partial \theta} = 0.$$

Here we find that $mr^2\dot{\theta}$, the *angular momentum*, is conserved as the Lagrangian does not depend on θ.

3.4 Lagrangian of a free particle and Gallilean invariance

Finally, let us consider a simple system where the transformation $\mathbf{q} = \mathbf{q}(\mathbf{x}, t)$ explicitly depends on time. The simplest form of such a transformation is a Gallilean transformation

$$x' = x + Vt \tag{39}$$

We consider the Lagrangian of a free particle:

$$L = \frac{1}{2}m\vec{v}^2 \tag{40}$$

Let us perform a Gallilean transformation $\vec{v}' = \vec{v} + \vec{V}$. Naively, the form of the Lagrangian 40 changes:

$$L(v') = \frac{1}{2}m\Big(\vec{v}^2 + \vec{V}^2 + 2(\vec{v}\cdot\vec{V})\Big) \tag{41}$$

Notice, however that the additional terms form a total derivative:

$$\vec{V}^2 + 2(\vec{v}\cdot\vec{V}) = \frac{d}{dt}\Big(\vec{V}^2 t + 2(\vec{r}\cdot\vec{V})\Big) \tag{42}$$

and therefore, as discussed around Eq. (32) does not change the extremum of the action.

References

[1] Analytical Mechanics, G.R. Fowles and G.L. Cassiday 6th or 7th edition (Thomson Learning, inc., 1999), ISBN 9780534408138.