Physica **108A** (1981) 557–566 *North-Holland Publishing Co.*

A CANONICAL TRANSFORMATION RELATING THE LAGRANGIAN AND EULERIAN DESCRIPTION OF IDEAL HYDRODYNAMICS

W. VAN SAARLOOS

Instituut-Lorentz, Rijksuniversiteit te Leiden, Nieuwsteeg 18, 2311 SB Leiden, The Netherlands

Received 12 March 1981

We derive by means of a canonical transformation the variational principle of Seliger and Whitham for the Eulerian description of ideal hydrodynamics from the more familiar variational principle that yields the equations of motion of an ideal fluid in the Lagrangian description.

1. Introduction

In fluid mechanics, two ways exist to specify the fields. The one most often used is the Eulerian description, in which the fields are considered as functions of the position in space, and time. The Lagrangian description, on the other hand, is based on the observation that many quantities specifying the fluid refer more fundamentally to small identifiable pieces of matter, the "fluid particles". In the Lagrangian description one therefore considers all the fields as functions of the time and the label of the fluid particle, to which they pertain.

As the fluid equations for ideal flow in the Lagrangian specification reflect quite closely the equations of motion for ordinary point particles, it is not surprising that a variational principle for these equations is already known since and resembles the one for point-particles. In fact, the formulation of this variational principle is due to Lagrange himself. The formulation of a variational principle for the Eulerian way to specify the fields, however, has proceeded in steps and spanned a long period. Clebsch[1]) was the first to introduce in 1859 a variational principle for the case of incompressible flow. His analysis however applies to other cases of constrained flow as well, such as the case of compressible isentropic flow, which was investigated in detail by Bateman[2]) in 1929. It was only in 1968 that Seliger and Whitham extended Clebsch's variational principle to the most general case of compressible, nonisentropic flow. The Lagrangian they introduce is of a somewhat unexpected form; yet the corresponding Hamiltonian is just the energy and is therefore the same as the one found in the variational principle for the

0378-4371/81/0000–0000/$02.50

Lagrangian description, as one might expect. Hence the question forces itself upon us whether the variational principles for either of the two ways of specifying the fields are not in fact one and the same, so that they may be transformed into each other by means of a canonical transformation. It is the purpose of this paper to show that such a canonical transformation actually exists, and that it enables us to derive Seliger and Whitham's[3]) variational principle from the more familiar variational principle of Lagrange. Our derivation will clarify the variational principle of Seliger and Whitham, in that the so called Clebsch representation and the new fields, introduced by Seliger and Whitham in order to arrive at a proper variational principle, will turn out to emerge in a natural way.

The plan of this paper is to treat the variational principle for the Lagrangian description first in section 2. We then discuss the canonical transformation that will yield the variational principle for the Eulerian description in section 3.

2. Variational principle for the Lagrangian description

In the Lagrangian description of ideal fluid mechanics, one brings out the fact that the fluid motion may be viewed as the motion of small identifiable fluid particles. Accordingly one considers the positions $\boldsymbol{q}$ of these fluid particles as a function of the time. If we label the fluid elements with their Cartesian coordinates $\boldsymbol{m} = (m_1, m_2, m_3)$ at some initial time t_0 (which we will take equal to zero), the positions $\boldsymbol{q}$ thus become functions of $\boldsymbol{m}$ and t. Other fields are given as functions of $\boldsymbol{m}$ and t as well.

Before we discuss the equations of motion of an ideal fluid in the Lagrange description, it is instructive to realize that the transition from the Lagrangian to the Eulerian description can be viewed as a time-dependent coordinate-transformation. For, in the Eulerian description, the fields are given as a function of a fixed position in space and of the time. A fixed position in space, however, can not only be given by the three Cartesian coordinates r_1, r_2 and r_3 of the position vector $\boldsymbol{r}$, but also by the label of the fluid particle which is at that moment at $\boldsymbol{r}$. This is expressed by the relation

$$\boldsymbol{r} = \boldsymbol{q}(\boldsymbol{m}, t), \tag{1}$$

or, equivalently,

$$\boldsymbol{m} = \tilde{\boldsymbol{q}}(\boldsymbol{r}, t), \tag{2}$$

where $\tilde{\boldsymbol{q}}$ is the inverse of the function $\boldsymbol{q}$. Eqs. (1) and (2) show that for the specification of fixed positions in space, $\boldsymbol{m}$ may be viewed as a set of non-Cartesian, time-dependent coordinates.

For the discussion of the fluid equations in the Lagrangian description, we will need the Jacobian

$$J(m, t) = \frac{\partial(q_1, q_2, q_3)}{\partial(m_1, m_2, m_3)}, \tag{3}$$

and J_{ij}, the co-factor of $\partial q_i/\partial m_j$ in J. We also note the useful identity

$$\left(\frac{\partial \tilde{q}_i(r, t)}{\partial r_j}\right)_{r=q(m,t)} = J_{ji}(m, t)/J(m, t) \quad \text{(Cramer's rule)} \tag{4}$$

and the fact the gradient operator may be written as

$$\nabla_i \equiv \frac{\partial}{\partial r_i} = J^{-1}(m, t)\sum_{j=1}^{3} J_{ij}(m, t)\frac{\partial}{\partial m_j}. \tag{5}$$

Let us consider a fluid element with mass $\rho_0(m)\,\mathrm{d}m_1\,\mathrm{d}m_2\,\mathrm{d}m_3$, where $\rho_0(m)$ is the mass density at $t = 0$. In the Lagrangian description, the equation of motion for such a fluid element is given by Newton's law as

$$\rho_0(m)\ddot{q}_i(m, t) = -\sum_j J_{ij}\frac{\partial p(m, t)}{\partial m_j} = -J(m, t)\nabla_i p(m, t). \tag{6}$$

Here partial derivatives with respect to time are indicated by a dot. In order to get a closed description, the pressure p, which is a function of the mass-density ρ and the entropy per unit mass s, must be known. For ideal fluid motion, only the volume of a fluid element changes, whereas its mass and entropy remain the same. One thus has for the density

$$\rho(m, t) = \rho_0(m)/J(m, t). \tag{7}$$

and for the entropy per unit mass

$$s(m, t) = s_0(m). \tag{8}$$

Eqs. (6), (7) and (8), together with the function $p(\rho, s)$ form a closed set of equations.

The variational principle leading to eq. (6) is

$$\delta\int \mathrm{d}m \int_{t_1}^{t_2} \mathrm{d}t\, \mathscr{L}(q(m, t)) \equiv \delta\int \mathrm{d}m \int_{t_1}^{t_2} \mathrm{d}t\,[\rho_0(m)\dot{q}^2(m, t) - \rho_0(m)u(\rho(m, t), s_0(m))] = 0. \tag{9}$$

This equation should hold for arbitrary variations δq that vanish at t_1 and t_2 and at the boundary of the region of integration. In eq. (9), the internal energy

per unit mass u is a function of ρ and s, with derivatives

$$\left(\frac{\partial u}{\partial \rho}\right)_s = p/\rho^2, \tag{10}$$

$$\left(\frac{\partial u}{\partial s}\right)_\rho = T, \tag{11}$$

where T is the temperature.

As $\rho(\boldsymbol{m}, t)$ depends on the derivatives $\partial q_i/\partial m_j$ through eq. (7), the Euler–Lagrange equations corresponding to eq. (9) are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}_i}\right) + \sum_j \frac{\partial}{\partial m_j}\left(\frac{\partial \mathscr{L}}{\partial (\partial q_i/\partial m_j)}\right) = 0. \tag{12}$$

By using the explicit form of $\mathscr{L}$ and the thermodynamic relation (10) this equation becomes

$$\rho_0(\boldsymbol{m})\ddot{q}_i(\boldsymbol{m}, t) - \sum_j \frac{\partial}{\partial m_j}\left[\frac{\rho_0(\boldsymbol{m}) p(\boldsymbol{m}, t)}{\rho^2(\boldsymbol{m}, t)} \frac{\partial \rho(\boldsymbol{m}, t)}{\partial (\partial q_i/\partial m_j)}\right] = 0. \tag{13}$$

From eq. (7) we obtain

$$\frac{\partial \rho(\boldsymbol{m}, t)}{\partial (\partial q_i/\partial m_j)} = -\frac{\rho_0(\boldsymbol{m})}{J^2(\boldsymbol{m}, t)} \frac{\partial J(\boldsymbol{m}, t)}{\partial (\partial q_i/\partial m_j)} = -\frac{\rho^2(\boldsymbol{m}, t) J_{ij}(\boldsymbol{m}, t)}{\rho_0(\boldsymbol{m})}. \tag{14}$$

Upon substitution of eq. (14) into the Euler–Lagrange equation (13), one gets

$$\rho_0(\boldsymbol{m})\ddot{q}_i(\boldsymbol{m}, t) = -\sum_j \frac{\partial}{\partial m_j}(p(\boldsymbol{m}, t) J_{ij}(\boldsymbol{m}, t)). \tag{15}$$

With the aid of the identity

$$\sum_j \frac{\partial J_{ij}}{\partial m_j} = 0, \tag{16}$$

which may easily be proved using eq. (4), eq. (15) finally reduces to

$$\rho_0(\boldsymbol{m})\ddot{q}_i(\boldsymbol{m}, t) = -\sum_j J_{ij}(\boldsymbol{m}, t) \frac{\partial p(\boldsymbol{m}, t)}{\partial m_j}. \tag{17}$$

Comparison of eqs. (6) and (17) shows that the variational principle (9) is indeed consistent with the equations of motion for the ideal fluid in the Lagrangian description.

The transition to a Hamilton-formalism can be accomplished in the standard way. In terms of the generalized coördinate*) $\boldsymbol{q}$ and generalized momen-

* In order to contrast the *field* $\boldsymbol{q}(\boldsymbol{m}, t)$ with the vectors $\boldsymbol{r}$, which denote points in a Cartesian coordinate system, we call $\boldsymbol{q}(\boldsymbol{m}, t)$ a generalized coordinate and $\boldsymbol{r}$ a position-vector with position-coordinates r_1, r_2 and r_3.

tum $\boldsymbol{p}$, defined by

$$\boldsymbol{p}(\boldsymbol{m}, t) \equiv \frac{\partial \mathscr{L}}{\partial \dot{\boldsymbol{q}}(\boldsymbol{m}, t)} = \rho_0(\boldsymbol{m}) \dot{\boldsymbol{q}}(\boldsymbol{m}, t), \tag{18}$$

the Hamiltonian H reads

$$H \equiv \int \mathrm{d}\boldsymbol{m}\, \mathscr{H}(\boldsymbol{q}(\boldsymbol{m}, t), \boldsymbol{p}(\boldsymbol{m}, t)) \equiv \int \mathrm{d}\boldsymbol{m} \left[\tfrac{1}{2} \frac{\boldsymbol{p}^2(\boldsymbol{m}, t)}{\rho_0(\boldsymbol{m})} + \rho_0(\boldsymbol{m}) u(\rho(\boldsymbol{m}, t), s_0(\boldsymbol{m})) \right]. \tag{19}$$

The Hamilton equations read

$$\dot{\boldsymbol{q}}(\boldsymbol{m}, t) = \frac{\delta H}{\delta \boldsymbol{p}(\boldsymbol{m}, t)}, \quad \dot{\boldsymbol{p}}(\boldsymbol{m}, t) = -\frac{\delta H}{\delta \boldsymbol{q}(\boldsymbol{m}, t)}. \tag{20}$$

The functional derivatives used here are defined in the standard way (see e.g. Goldstein[4])). Using the explicit form of the Hamiltonian, these Hamilton equations become

$$\dot{q}_i(\boldsymbol{m}, t) = \frac{p_i(\boldsymbol{m}, t)}{\rho_0(\boldsymbol{m})}, \tag{21}$$

$$\dot{p}_i(\boldsymbol{m}, t) = \sum_j \frac{\partial}{\partial m_j} \left[\left(\frac{\partial u}{\partial \rho} \right)_s \frac{\partial \rho(\boldsymbol{m}, t)}{\partial (\partial q_i / \partial m_j)} \right]. \tag{22}$$

These equations are equivalent to the Euler–Lagrange equations (13); as was already discussed, eqs. (13) are in turn equivalent to the equations of motion, eq. (6).

We notice that the equation of motion for the fluid particles, eq. (6), resembles quite closely Newton's equation for point particles. Accordingly, the variational principle discussed above is completely analogous to the variational principle for point-particles. For we have seen that the Lagrangian just assumes its familiar form, namely "kinetic energy minus internal (potential) energy". Moreover, we remark that the Hamilton equations for $\dot{\boldsymbol{q}}$ (eq. (21)) is the same as the equation defining the generalized momentum $\boldsymbol{p}$; this one usually finds too. These analogies will be lost if we perform a transformation to the Eulerian description, as will be discussed in the next section.

3. Canonical transformation to Eulerian description

Instead of using the Lagrangian description, in which case one specifies fields with the label of the fluid particle to which they pertain, it is usually more convenient to employ the Eulerian specification of the fields. In the latter case, one considers the fields as functions of their position in space $\boldsymbol{r}$

and time. In the Eulerian description, the fluid flow is completely characterized by the five physical fields, the density $\rho(r, t)$, the velocity $v(r, t)$ and the entropy $s(r, t)$.

We have already discussed at the beginning of section 2 that the transition from the Lagrangian to the Eulerian description and vice versa may be viewed as a time-dependent coordinate-transformation. This transformation, which is expressed by eqs. (1) and (2), enables us to relate the density and entropy in the two types of specifications as

$$\rho(r, t)\big|_{r=q(m,t)} = \rho(m, t), \quad \text{or} \quad \rho(r, t) = \rho(m, t)\big|_{m=\tilde{q}(r,t)}, \tag{23}$$

$$s(r, t)\big|_{r=q(m,t)} = s_0(m), \quad \text{or} \quad s(r, t) = s_0(m)\big|_{m=\tilde{q}(r,t)}. \tag{24}$$

As the velocity at the position r at time t is just equal to the velocity of the fluid element which is at r at that particular moment, we also have

$$v(r, t)\big|_{r=q(m,t)} = \frac{\partial q(m, t)}{\partial t}, \quad \text{or} \quad v(r, t) = \frac{\partial q(m, t)}{\partial t}\Big|_{m=\tilde{q}(r,t)}. \tag{25}$$

We will now show that the variational principle discussed in the previous section can be transformed by means of a canonical transformation into a variational principle for the fields $\rho(r, t)$, $v(r, t)$ and $s(r, t)$. In particular, we will look for a Hamiltonian formalism in which ρ, ρs ($\equiv s_v$, the entropy per unit volume) and the function $\rho\tilde{q}_3$ play the role of the new generalized momenta $P_i(r, t)$.

To be specific, we will consider a generating functional $F([q(m, t)], [Q(r, t)])$, where $Q(r, t)$ is the new generalized coordinate. The theory of canonical transformations for fields, which e.g. has been discussed in detail by Kobussen and Broer[5,6]), proceeds along the same lines as the familiar theory of canonical transformations for point-particles (see e.g. Goldstein[4])). For a generating functional of the above form, the generalized coordinates $p(m, t)$ and $P(r, t)$ are given in terms of $q(m, t)$ and $Q(r, t)$ by

$$p_i(m, t) = \frac{\delta F}{\delta q_i(m, t)}, \tag{26}$$

$$P_i(r, t) = -\frac{\delta F}{\delta Q_i(r, t)}, \tag{27}$$

whereas the Hamiltonian is invariant since the generating functional F does not depend explicitly on time.

The following form turns out to be appropriate

$$F = -\int dr \int dm[\{\rho_0(m)Q_1(r, t) + \rho_0(m)s_0(m)Q_2(r, t) + \rho_0(m)m_3Q_3(r, t)\}\delta(r - q(m, t))]. \tag{28}$$

We will first evaluate eq. (27). By making use of the identity

$$\delta(\boldsymbol{r} - \boldsymbol{q}(\boldsymbol{m}, t)) = J^{-1}(\boldsymbol{m}, t)\delta(\boldsymbol{m} - \tilde{\boldsymbol{q}}(\boldsymbol{r}, t)), \tag{29}$$

we obtain with the aid of eqs. (7) and (23)

$$P_1(\boldsymbol{r}, t) = -\frac{\delta F}{\delta Q_1(\boldsymbol{r}, t)} = \frac{\rho_0(\boldsymbol{m})}{J(\boldsymbol{m}, t)}\bigg|_{\boldsymbol{m}=\tilde{\boldsymbol{q}}(\boldsymbol{r}, t)} = \rho(\boldsymbol{m}, t)|_{\boldsymbol{m}=\tilde{\boldsymbol{q}}(\boldsymbol{r}, t)} = \rho(\boldsymbol{r}, t). \tag{30}$$

Similarly we get

$$P_2(\boldsymbol{r}, t) = \rho(\boldsymbol{r}, t)s(\boldsymbol{r}, t) = s_v(\boldsymbol{r}, t), \tag{31}$$

$$P_3(\boldsymbol{r}, t) = \rho(\boldsymbol{r}, t)\tilde{q}_3(\boldsymbol{r}, t). \tag{32}$$

Indeed the mass density and the entropy density are two of the new generalized momenta. We will comment on eq. (32) later.

Next we evaluate eq. (26). By first performing the $\boldsymbol{r}$-integration in the expression (28), we obtain

$$\begin{aligned}
\boldsymbol{p}(\boldsymbol{m}, t) = &-\frac{\delta}{\delta q_i(\boldsymbol{m}, t)}\int \mathrm{d}\boldsymbol{m}[\rho_0(\boldsymbol{m})Q_1(\boldsymbol{r}, t)\,|_{\boldsymbol{r}=\boldsymbol{q}(\boldsymbol{m}, t)} + \rho_0(\boldsymbol{m})s_0(\boldsymbol{m})Q_2(\boldsymbol{r}, t)\,|_{\boldsymbol{r}=\boldsymbol{q}(\boldsymbol{m}, t)} \\
&+ \rho_0(\boldsymbol{m})m_3Q_3(\boldsymbol{r}, t)\,|_{\boldsymbol{r}=\boldsymbol{q}(\boldsymbol{m}, t)}] \\
= &-\rho_0(\boldsymbol{m})\frac{\partial Q_1(\boldsymbol{r}, t)}{\partial \boldsymbol{r}}\bigg|_{\boldsymbol{r}=\boldsymbol{q}(\boldsymbol{m}, t)} - \rho_0(\boldsymbol{m})s_0(\boldsymbol{m})\frac{\partial Q_2(\boldsymbol{r}, t)}{\partial \boldsymbol{r}}\bigg|_{\boldsymbol{r}=\boldsymbol{q}(\boldsymbol{m}, t)} \\
&- \rho_0(\boldsymbol{m})m_3\frac{\partial Q_3(\boldsymbol{r}, t)}{\partial \boldsymbol{r}}\bigg|_{\boldsymbol{r}=\boldsymbol{q}(\boldsymbol{m}, t)}.
\end{aligned} \tag{33}$$

By using eq. (21) and the identification (25), eq. (33) can be rewritten in a more transparent way, viz.

$$\boldsymbol{v}(\boldsymbol{r}, t) = -\nabla Q_1(\boldsymbol{r}, t) - s(\boldsymbol{r}, t)\nabla Q_2(\boldsymbol{r}, t) - \tilde{q}_3(\boldsymbol{r}, t)\nabla Q_3(\boldsymbol{r}, t). \tag{34}$$

This is the form of the so-called Clebsch representation* for the velocity field which was first used by Seliger and Whitham[3]).

If we write the Hamiltonian as a functional of the new fields with the aid of eqs. (30)–(33), the new Hamilton density turns out to be equal to the energy

* Clebsch[1]) introduced his way of representing the velocity field for the case that the term $s\nabla Q_2$ may be omitted.

density in the Eulerian specification,

$$H = \int \mathrm{d}r\, \mathscr{H}(\boldsymbol{Q}(r,t), \boldsymbol{P}(r,t)),$$

$$= \int \mathrm{d}r \left[\left(\sum_i P_i(r,t) \nabla Q_i(r,t) \right)^2 \Big/ 2P_1(r,t) + P_1(r,t) u\left(P_1(r,t), \frac{P_2(r,t)}{P_1(r,t)} \right) \right],$$

$$= \int \mathrm{d}r [\tfrac{1}{2}\rho(r,t) v^2(r,t) + \rho(r,t) u(\rho(r,t), s(r,t))]. \tag{35}$$

The variational principle for the Eulerian description of ideal fluids that we derived thus consists of the Hamiltonian (35), together with the Clebsch representation (34) and the identifications (30)–(32). The corresponding Hamilton equations are (since from now on all functions depend on r and t, we will not write these arguments explicitly)

$$\dot{Q}_i = \frac{\delta H}{\delta P_i} \rightarrow \begin{cases} \dot{Q}_1 = -\boldsymbol{v} \cdot \nabla Q_1 - \tfrac{1}{2}\boldsymbol{v}^2 + \mu, & (36) \\ \dot{Q}_2 = -\boldsymbol{v} \cdot \nabla Q_2 + T, & (37) \\ \dot{Q}_3 = -\boldsymbol{v} \cdot \nabla Q_3, & (38) \end{cases}$$

$$\dot{P}_i = -\frac{\delta H}{\delta Q_i} \rightarrow \begin{cases} \dot{P}_1 = -\nabla \cdot P_1 \boldsymbol{v}, \quad \text{or } \dot{\rho} = -\nabla \cdot \rho \boldsymbol{v}, & (39) \\ \dot{P}_2 = -\nabla \cdot P_2 \boldsymbol{v}, \quad \text{or } \dot{s}_v = -\nabla \cdot s_v \boldsymbol{v}, & (40) \\ \dot{P}_3 = -\nabla \cdot P_3 \boldsymbol{v}. & (41) \end{cases}$$

In eq. (36) we introduced the chemical potential per unit mass, defined by

$$\mu = u - Ts + p/\rho = \left(\frac{\partial \rho u}{\partial \rho} \right)_{s_v}. \tag{42}$$

Eqs. (39) and (40) are just the well-known mass and entropy conservation equations. With the aid of eq. (39) and the identification (32), eqs. (40) and (41) may also be written in the form

$$\frac{\mathrm{d}s}{\mathrm{d}t} = 0, \tag{43}$$

$$\frac{\mathrm{d}\tilde{q}_3}{\mathrm{d}t} = 0, \tag{44}$$

where the total time derivative $\mathrm{d}/\mathrm{d}t$ is defined by

$$\frac{\mathrm{d}}{\mathrm{d}t} \equiv \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla. \tag{45}$$

Eq. (43) expresses the fact that if we follow a fluid element, its entropy per unit mass remains constant. In view of the relation $m_3 = \tilde{q}_3(r, t)$, we can similarly conclude that eq. (44) gives expression to the fact that the third label also remains constant in following a fluid particle*.

For the equation of motion for the velocity-field we obtain with the aid of eqs. (34), (43) and (44)

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\nabla\frac{\mathrm{d}Q_1}{\mathrm{d}t} - s\nabla\frac{\mathrm{d}Q_2}{\mathrm{d}t} - \tilde{q}_3\nabla\frac{\mathrm{d}Q_3}{\mathrm{d}t} + (\nabla\boldsymbol{v})\cdot(\nabla Q_1 + s\nabla Q_2 + \tilde{q}_3\nabla Q_3). \tag{46}$$

The fourth term in the right-hand side of eq. (46), which is due to the fact that the gradient operator and the operator $\mathrm{d}/\mathrm{d}t$ do not commute, is equal to $-\frac{1}{2}\nabla\boldsymbol{v}^2$ (cf. eq. (34)). By substituting eqs. (36)–(38) into eq. (46), and using also the Gibbs–Duhem relation, we finally get

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\nabla\mu - s\nabla T = -\frac{1}{\rho}\nabla p. \tag{47}$$

Hence the Hamilton equations (36)–(41) indeed yield the Euler equation for the velocity-field. We have thus shown that the canonical transformation we introduced yields a variational principle, from which the well-known equations for $\rho(r, t)$, $v(r, t)$ and $s(r, t)$ follow. In addition, a sixth equation is found, which expresses the fact that the third label is conserved.

Let us retrace the origin of the fact that Q_3 is also conserved (cf. eq. (38)), i.e. remains constant if we follow a fluid particle. In the Lagrangian description the internal energy depends on m_1, m_2 and m_3 only via the *two* known functions $\rho_0(\boldsymbol{m})$ and $s_0(\boldsymbol{m})$ (cf. eq. (9)). Consequently, there is then a gauge-freedom, namely a freedom to choose, say, the third label m_3. This gauge-freedom is still present in the Eulerian description, in that u does not depend on $\tilde{q}_3$. This in turn implies that Q_3 is conserved, and we therefore conclude that the fact that Q_3 is conserved is related to the gauge-freedom.

From the Hamiltonian density (35), the Lagrangian of Seliger and Whitham[3]) may be obtained. The Lagrangian density and Hamiltonian density are related by

$$\mathscr{L} = \sum_i P_i\dot{Q}_i - \mathscr{H}. \tag{48}$$

If we use the identifications (30)–(32) to write the Lagrangian density in this

* We remark that a canonical transformation, for which the three functions $\tilde{q}_i$ become the new generalized coordinates, was discussed by Kobussen and Broer[5,7]). Although in their case the three equations expressing conservation of label are found, no equations for ρ and s are then obtained.

equation as a function of ρ, s, $\tilde{q}_3$, Q_1, Q_2 and Q_3, we obtain

$$\mathscr{L} = \rho[\dot{Q}_1 + s\dot{Q}_2 + \tilde{q}_3\dot{Q}_3 - \tfrac{1}{2}v^2 - u(\rho, s)]. \tag{49}$$

This Lagrangian density, together with the Clebsch representation (34), was introduced some years ago by Seliger and Whitham[3]). One easily checks that the Euler–Lagrange equations corresponding to it are equivalent to the Hamilton equations (36)–(41).

It was originally suggested by Lin[8]), that equations, expressing label conservation, should be added to the hydrodynamic equations for ρ, v and s, in order to be able to arrive at a proper variational principle. Later, Seliger and Whitham[3]) argued that a variational principle can be formulated for the hydrodynamic equations in the Eulerian description if they are supplemented with an equation that expresses the conservation of only one of the labels. Accordingly, they introduced the field $\tilde{q}_3$ and arrived at the Lagrangian density (49). In view of the fact that our derivation fully supports the interpretation of $\tilde{q}_3$ as suggested by Lin, we may conclude that Lin has essentially made the right point.

We finally remark that Poisson-brackets for the hydrodynamic fields may be introduced with the aid of the generalized coordinates and momenta given in this section. This is discussed in a recent paper by the author and Bedeaux and Mazur[9]).

Acknowledgements

It is a pleasure to thank Professors D. Bedeaux and P. Mazur for illuminating discussions on the subject of this paper and for a critical reading of the manuscript. A stimulating discussion with Dr. J.M. Rubí is also gratefully acknowledged.

References

1) A. Clebsch, Crelle (J. Reine Angew. Math.) **56** (1859) 1.
2) H. Bateman, Proc. Roy. Soc. **A125** (1929) 598.
3) R.L. Seliger and G.B. Whitham, Proc. Roy. Soc. **A305** (1968) 1.
4) H. Goldstein, Classical Mechanics (Addison Wesley, New York 1977).
5) J.A. Kobussen, Some Methods of Classical Mechanics applied to Continuous Systems, Thesis, University of Eindhoven, (1973).
6) L.J.F. Broer and J.A. Kobussen, Physica **61** (1972) 275.
7) L.J.F. Broer and J.A. Kobussen, Appl. Sci. Res. **29** (1974) 419.
8) See footnote 2 in J. Serrin, Handbuch der Physik, vol. 8/1, (1959) p. 148, and C.C. Lin, Liquid Helium, Proc. Int. School of Physics, (Macmillan, New York, 1963).
9) W. van Saarloos, D. Bedeaux and P. Mazur, Physica **107A** (1981) 109.