Comment on "Absolute and Convective Instabilities in Nonlinear Systems"

When a spatially extended system goes unstable, the ensuing dynamics depends sensitively on whether the system is convectively unstable [in which case perturbations grow in time but are convected away fast enough that they die at each fixed position in the (lab) frame considered] or absolutely unstable (in which case there exists a perturbation and a location where the perturbation does not decay). The distinction between the two cases for infinitesimal disturbances is well understood; such a *linear* stability analysis captures most of the essential physics near a supercritical (continuous) bifurcation. Recently, Chomaz [1] studied the *nonlinear* convective (NLC) versus absolute (NLA) instability near a subcritical (discontinuous) bifurcation for a simple equation that derives from a free-energy-like (Lyapunov) function. The purpose of this Comment is to point out that the case studied by Chomaz is quite restrictive, since it relies on the existence of a unique front separating the basic state from the bifurcating state. In the general case there is a continuum of bifurcating states and an ensuing continuum of fronts, so the problem of *selection* must be faced. The situation was discussed earlier by two of us [2] in a general investigation of front and pulse propagation near subcritical bifurcations. The extension to systems not governed by a Lyapunov function is particularly relevant for the study of nonlinear stability of open hydrodynamic flows or of systems with traveling waves.

As a simple model for dynamics near a subcritical bifurcation, Chomaz [1] studied the real equation

$$\partial_t A + U_0 \partial_x A = \partial_x^2 A + \mu A + A^3 - A^5 . \tag{1}$$

The nonlinear stability properties depend on the response to disturbances of *finite* extent and amplitude. For $-\frac{1}{4} < \mu < 0$ Eq. (1) admits two homogeneous stable states, $A_0 = 0$ and $A_2 \neq 0$. To study the nonlinear stability of the A_0 state it suffices to consider a front solution joining the state A_2 for $x \to -\infty$ with the state A_0 for $x \to \infty$, in the symmetrical ($U_0 = 0$) frame where the $U_0 \partial_x A$ term is absent. If the front speed v of this solution is negative, an isolated droplet of the A_2 state in a background of the A_0 state shrinks; hence the A_0 state is stable. If v is positive, A_2 droplets grow and the A_0 state is (nonlinearly) unstable. Since for $U_0 = 0$, Eq. (1) is governed by a Lyapunov function [$\partial_t A = -\delta \mathcal{L}/\delta A$, $\mathcal{L} = \int dx \{(\partial_x A)^2/2 - \mu A^2/2 - A^4/4 + A^6/6\}$], the sign of v depends on the relative magnitude of $\mathcal{L}(A_0)$ and $\mathcal{L}(A_2)$, and $v = 0$ for $\mu = \mu_M = -\frac{3}{16}$ where $\mathcal{L}(A_0) = \mathcal{L}(A_2)$. In the unstable domain $\mu > \mu_M$ the instability in the U_0 frame is convective (NLC) for $v - U_0 < 0$, and absolute (NLA) for $v - U_0 \geq 0$.

When a Hopf bifurcation to traveling waves occurs, the amplitude dynamics near a subcritical bifurcation can be modeled by an extension of (1), the complex Ginzburg-Landau equation, which in the symmetrical ($U_0 = 0$) frame reads

$$\partial_t A = (1 + ic_1)\partial_x^2 A + \mu A + (1 + ic_3) A|A|^2 + (-1 + ic_5) A |A|^4 . \tag{2}$$

Here A is the complex valued amplitude, and the c's are real parameters associated with the linear (c_1) and nonlinear (c_3, c_5) dispersion. Equation (2) cannot be derived from a Lyapunov function, and contrary to (1) has a continuum of bifurcating states.

The surprising finding of Ref. [2] is that the stability properties of the state A_0 are largely determined by the existence or absence of an exact *nonlinear* front solution with speed $v^\dagger(\mu, c_1, c_3, c_5)$ that increases for increasing μ and is zero for $\mu = \mu_3(c_1, c_3, c_5)$. It is found [2] that either (a) this front solution exists and has positive $v^\dagger$ for some range $\mu > \mu_3$ with $\mu_3 < 0$; (b) for all $\mu < 0$ the front speed is negative (i.e., $\mu_3 > 0$); or (c) for $\mu < 0$ no nonlinear front solution exists.

In case (a) the behavior for $\mu > \mu_3$ is similar to that found in the real equation when $\mu > \mu_M$: The state A_0 is unstable, and the instability is NLA for $v^\dagger - U_0 \geq 0$ and NLC for $v^\dagger - U_0 < 0$. For $\mu < \mu_3$, on the other hand, typically stationary pulse solutions exist, over a range $\mu_2 < \mu < \mu_3$, so although $v^\dagger < 0$, the state A_0 remains *unstable.* Since the pulse velocity is in general zero, the instability is NLC for any $U_0 > 0$. For $\mu < \mu_2$ the state A_0 is stable. In case (b) the pulse region extends up to $\mu = 0$, and for $\mu > 0$ the stability properties are similar to those of a supercritical bifurcation with a front velocity $v \propto \sqrt{\mu}$. For case (c) less is known, but chaotically spreading front solutions as well as pulses have been found [2]. In some experiments [3], the latter structures help stabilize a system by absorbing small perturbations that are convected into them. It is an open question which regime is relevant for planar Poiseuille flow, where $c_1 \approx 0.4$ and $c_3 \approx 6$ [4] but c_5 is not known.

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[1] J. M. Chomaz, Phys. Rev. Lett. **69**, 1931 (1991).
[2] W. van Saarloos and P. C. Hohenberg, Phys. Rev. Lett. **64**, 743 (1990); Physica (Amsterdam) **56D**, 303 (1992).
[3] See, e.g., P. Kolodner, Phys. Rev. Lett. **66**, 1165 (1991); F. Daviaud, J. Hegseth, and P. Bergé, Phys. Rev. Lett. **69**, 2511 (1992).
[4] L. M. Hocking, K. Stewartson, and J. T. Stuart, J. Fluid Mech. **51**, 705 (1972).

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